MOMENT MAP EQUATIONS IN GAUGE THEORY AND COMPLEX GEOMETRY

Lecture 1

Canonical metrics on bundles and stability

Oscar García-Prada
ICMAT-CSIC, Madrid

Cargèse, 25 June 2024

Connections, curvature and metrics on vector bundles

- Let M be a **smooth manifold** and $E \to M$ be a **smooth complex vector bundle**. A **connection** on E is a $\mathbb{C}$-linear map

$$D : \Omega^0(M,E) \longrightarrow \Omega^1(M,E)$$

satisfying

$$D(fs) = df \otimes s + fDs \quad \text{for} \quad s \in \Omega^0(M,E) \quad \text{and} \quad f \in \Omega^0(M)$$

- This can be extended to a $\mathbb{C}$-linear map

$$D : \Omega^i(M,E) \longrightarrow \Omega^{i+1}(M,E)$$

satifying

$$D(\alpha s) = d\alpha \otimes s + (-1)^i \alpha Ds \quad \text{for} \quad s \in \Omega^0(M,E) \quad \text{and} \quad \alpha \in \Omega^i(M).$$

- The **curvature** of D is defined as

$$F_D = D^2 \in \Omega^2(\operatorname{End} E).$$

  Hence, the curvature is the **obstruction** for D to define a complex $\Omega^0(M,E) \xrightarrow{D} \Omega^1(M,E) \xrightarrow{D} \Omega^2(M,E) \xrightarrow{D} \cdots$

- A **Hermitian metric** h on E is a smooth assignment of a Hermitian product to each fibre of E.
- A connection D on E is said to be **unitary** (or **Hermitian**) if

$$dh(s,t) = h(Ds,t) + h(s,Dt) \ \text{ for } \ s,t \in \Omega^0(M,E).$$

- If D is a **unitary connection**, then

$$F_D \in \Omega^2(M, \operatorname{End}(E,h)),$$

  where $\operatorname{End}(E,h)$ is the **bundle of skew-Hermitian endomorphisms** of (E,h).

- Let M be now a **complex manifold**.
- We will look now at **holomorphic vector bundles** from the point of view of $\bar\partial$-operators.
- Let $\mathbb{E} \to M$ be a smooth complex vector bundle. A $\bar\partial$**-operator** (or **Dolbeault operator**) on $\mathbb{E}$ is $\mathbb{C}$-linear map

$$\bar\partial_E : \Omega^0(M, \mathbb{E}) \longrightarrow \Omega^{0,1}(M, \mathbb{E}),$$

which satisfies

$$\bar\partial_E(fs) = \bar\partial fs + f\bar\partial_E s, \quad \text{for } f \in \Omega^0(M) \text{ and } s \in \Omega^0(M, \mathbb{E}).$$

This can be extended to a $\mathbb{C}$-linear map

$$\bar\partial_E : \Omega^{0,i}(M, \mathbb{E}) \longrightarrow \Omega^{0,i+1}(M, \mathbb{E})$$

satifying

$$\bar\partial_E(\alpha s) = \bar\partial\alpha \otimes s + (-1)^i \alpha \bar\partial_E s \quad \text{for } s \in \Omega^0(M, \mathbb{E}) \text{ and } \alpha \in \Omega^{0,i}(M).$$

- A **holomorphic structure** on a smooth complex vector bundle $\mathbb{E} \to M$ is a $\bar\partial$-operator $\bar\partial_E$ on $\mathbb{E}$ such that

$$\bar\partial_E^2 = 0 \quad (\textbf{integrability})$$

Theorem

The pair $E = (\mathbb{E}, \bar\partial_E)$ with $\bar\partial_E^2 = 0$ is equivalent to a holomorphic vector bundle E (in the usual sense of having holomorphic transition functions).

- Of course, if M is a Riemann surface, $\Omega^{0,2}(M) = 0$, and hence the condition $\bar\partial_E^2 = 0$ is always satisfied.
- Let $\mathbb{E} \to M$ be a smooth complex vector bundle. Any connection D on $\mathbb{E}$ defines a $\bar\partial$-operator by the rule

$$\bar\partial_E = D^{0,1}.$$

We use here that $\Omega^1(M,\mathbb{E}) = \Omega^{1,0}(M,\mathbb{E}) \oplus \Omega^{0,1}(M,\mathbb{E})$.

- A $\bar{\partial}$-operator may come from many connections. However, one has the following important result.

Chern correspondence

Let h be a Hermitian metric on $\mathbb{E}$. Then, there is a **one-to-one** correspondence between $\bar{\partial}$-operators on $\mathbb{E}$ and unitary connections on $(\mathbb{E}, h)$.

- In particular, if $E \to M$ is a holomorphic vector bundle equipped with a Hermitian metric h, there is a unique h-unitary connection D compatible with the holomorphic structure, meaning that $\bar{\partial}_E = D^{0,1}$. This is called the **Chern connection** of (E, h), and its curvature will be denoted by $F_h = D^2$.

Holomorphic vector bundles over Riemann surfaces

- Let X be a **compact Riemann surface**, and let E be a **holomorphic vector bundle** over X.
- Let h be a **Hermitian metric** on E, D be the **Chern connection** and $F_h = D^2$ be its **curvature**.
- A natural condition to ask is that D be **flat**:

$$F_h = 0.$$

- From **Chern–Weil theory**, the first Chern class $c_1 E)$ is represented by

$$c_1(E, h) = \frac{i}{2\pi} \mathrm{Tr}(F_h)$$

  and hence $F_h = 0 \Longrightarrow c_1(E) = 0$.
- If $c_1(E) \neq 0$, then the closest to flatness that we can have is that the curvature be central. To formulate this condition, fix a metric metric on X, with Kähler form ω normalized such that $\mathrm{vol}(X) = \int_X \omega = 2\pi$.

Central curvature equation

We say that D has **central curvature** if

$$F_h = -i\mu I_E \omega, \tag{1}$$

where I_E is the identity endomorphism of E and μ is a real constant.

- Taking traces in (1) and integrating, after defining the **degree** of E as

$$\deg(E) := \int_X c_1(E,h) = \frac{i}{2\pi}\int_X \mathrm{Tr}(F_h),$$

we have that

$$\mu = \mu(E) = \deg E / \operatorname{rank} E,$$

where $\mu(E)$ is called the **slope** of E.

Holomorphic vector bundles over Riemann surfaces

- Want to study **conditions for existence of solutions** to (1).
- The first case to consider is $E = L$ a **line bundle**. In this situation a Hermitian metric on L can be written as $h = e^u$, where u is a **real function**, and
  Exercise 1.
  $$(1) \iff \Delta u = f, \quad \textbf{Laplace/Poisson equation}$$
  where f is a real function such that $\int_X f = 0$.
- In the line bundle case hence (1) can always be solved. The situation is very different in higher rank.
- A holomorhic vector bundle E is said to be **stable** if for every proper holomorphic subbundle $F \subset E$
  $$\mu(F) < \mu(E).$$
  This concept was introduced by **Mumford** and arises from **Geometric Invariant Theory** (GIT).

- In fact, stability can also be deduced from the differential geometry:
  Exercise 2. Let E be an indecomposable holomorphic vector bundle over X. Then

$$\text{existence of } h \text{ satisfying } (1) \Longrightarrow \text{stability of } E.$$

- **Hint**: Let $F \subset E$ be a holomorphic subbundle, and let $Q = E/F$. These fit in an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0.$$

  A Hermitian metric h on E defines a smooth splitting $E \cong F \oplus Q$, with respect to which

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_F & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix},$$

  where $\bar{\partial}_F$ and $\bar{\partial}_Q$ are the corresponding $\bar{\partial}$ operators on F and Q, respectively, and $\beta \in \Omega^{0,1}(X, \mathrm{Hom}(Q, F))$.

- Another concept which is relevant here is that of polystability. The vector bundle E is **polystable** if $E = \bigoplus E_i$, where E_i is stable and $\mu(E_i) = \mu(E)$.
- We have that if E is a holomorphic vector bundle over X (not necessarily indecomposable)

$$\textbf{existence of } h \text{ satisfying (1)} \Longrightarrow \textbf{polystability of } E.$$

In fact the converse is also true.

Theorem (Narasimhan–Seshadri, 1965)

A holomorphic vector bundle E admits a Hermitian metric with central curvature if and only if it is polystable.

- **Remark**. This is a reformulation of the theorem of Narasimhan–Seshadri due to **Atiyah–Bott** (1982), of which a proof was given by **Donaldson** (1982).

Moduli spaces of connections and holomorphic structures

- Let $\mathbb{E}$ be a smooth complex vector bundle over a compact Riemann surface X (equipped with a Kähler form ω as above) and h be a Hermitian metric on $\mathbb{E}$.
- Consider the following sets

$$\begin{aligned} \mathscr{D} &:= \{\text{connections } D \text{ on } \mathbb{E}\}, \\ \mathscr{A} &:= \{\text{unitary connections } d_A \text{ on } (\mathbb{E}, h)\} \subset \mathscr{D}, \\ \mathscr{C} &:= \{\text{holomorphic structures } \bar{\partial}_E \text{ on } \mathbb{E}\}. \end{aligned}$$

And the **gauge groups**

$$\begin{aligned} \mathscr{G}^c &:= \{\text{bundle automorphisms of } \mathbb{E}\}, \\ \mathscr{G} &:= \{\text{bundle automorphisms of } \mathbb{E} \text{ preserving } h\} \subset \mathscr{G}^c, \end{aligned}$$

- $\mathscr{G}^c$ acts on $\mathscr{D}$ and $\mathscr{C}$ respectively by the rule

$$g \cdot D = gDg^{-1} \text{ and } g \cdot \bar{\partial}_E = g\bar{\partial}_E g^{-1} \text{ for } g \in \mathscr{G}^c,\ D \in \mathscr{D},\ \bar{\partial}_E \in \mathscr{C}.$$

Moduli spaces of connections and holomorphic structures

- The action of $\mathscr{G}^c$ on $\mathscr{D}$ restricts to an action of the unitary gauge group $\mathscr{G} \subset \mathscr{G}^c$ on the set of unitary connections $\mathscr{A} \subset \mathscr{D}$.
- Consider the set of **central curvature unitary connections** on $(\mathbb{E}, h)$:

$$\mathscr{A}_0 := \{d_A \in \mathscr{A} \quad : \quad F_A := d_A^2 = -i\mu I_{\mathbb{E}}\omega\}$$

- Since $F_{g \cdot D} = gF_D g^{-1}$, the set $\mathscr{A}_0$ is invariant under the action of $\mathscr{G}$, and we can consider the **moduli space of central curvature unitary connections** on $(\mathbb{E}, h)$

$$\mathscr{A}_0/\mathscr{G}.$$

Moduli spaces of connections and holomorphic structures

- Consider now

$$\mathscr{C}^{ps} := \{\bar{\partial}_E \in \mathscr{C} \quad : \quad E = (\mathbb{E}, \bar{\partial}_E) \text{ is polystable}\}.$$

  The set $\mathscr{C}^{ps}$ is invariant under the action of $\mathscr{G}^c$ and we can consider the **moduli space of holomorphic structures** supported by $\mathbb{E}$

$$\mathscr{C}^{ps}/\mathscr{G}^c.$$

We can now reformulate the theorem of Narasimhan–Seshadri as follows.

Theorem (Narasimhan–Seshadri, 1965; Donaldson, 1982)

The Chern correspondence $\mathscr{A} \leftrightarrow \mathscr{C}$ induces a **bijection**

$$\mathscr{A}_0/\mathscr{G} \longleftrightarrow \mathscr{C}^{ps}/\mathscr{G}^c.$$

Under this bijection, **stable** holomorphic structures are in correspondence with **irreducible** connections.

Higgs bundles over Riemann surfaces

- Let X be a **compact Riemann surface** and K be its **canonical line bundle**.
- A **Higgs bundle** over X is a pair consisting of a holomorphic vector bundle $E \to X$, together with a sheaf homomorphism (the **Higgs field**) $\Phi : E \longrightarrow E \otimes K$, i.e.

$$\Phi \in H^0(X, \operatorname{End} E \otimes K).$$

- The Higgs bundle (E, Φ) is said to be **stable** if

$$\mu(F) < \mu(E)$$

  for every proper subbundle $F \subset E$ such that $\Phi(F) \subset F \otimes K$.
- The Higgs bundle (E, Φ) is **polystable** if $(E, \Phi) = \oplus (E_i, \Phi_i)$, where (E_i, Φ_i) is stable and $\mu(E_i) = \mu(E)$.

Hitchin equation

Let (E, Φ) be a Higgs bundle over X, equipped with a Kähler form ω as above. A natural condition to ask for a Hermitian metric h on E is that

$$F_h + [\Phi, \Phi^*] = -i\mu I_E \omega, \qquad (2)$$

- Here $[\Phi, \Phi^*] = \Phi\Phi^* + \Phi^*\Phi$ is the usual extension of the Lie bracket to Lie-algebra valued forms. Since $\mathrm{Tr}[\Phi, \Phi^*] = 0$, as in (1), $\mu = \mu(E)$.
- **Exercise 3**. Let (E, Φ) be a Higgs bundle over X. Then **existence** of h satisfying (2) $\Longrightarrow$ **polystability** of (E, Φ).

Theorem (Hitchin, 1987; Simpson, 1988)

A Higgs bundle (E, Φ) admits a Hermitian metric satisfying the Hitchin equation if and only if it is polystable.

Moduli spaces of Higgs bundles over Riemann surfaces

- To describe the moduli space of Higgs bundles in **differential-geometric** terms, let $\mathbb{E}$ be a smooth complex vector bundle over X, and consider the **set of pairs**

$$\mathscr{H} = \{(\bar\partial_E, \Phi) \in \mathscr{C} \times \Omega^{1,0}(X, \operatorname{End}\mathbb{E}) \quad : \quad \bar\partial_E \Phi = 0\}.$$

- The gauge group $\mathscr{G}^c$ acts on $\Omega^{1,0}(X, \operatorname{End}\mathbb{E})$ by

$$g \cdot \Phi = g\Phi g^{-1} \quad \text{where} \quad g \in \mathscr{G}^c \quad \text{and} \quad \Phi \in \Omega^{1,0}(X, \operatorname{End}\mathbb{E}),$$

  This, combined with the action on $\mathscr{C}$, gives an action on $\mathscr{H}$.
- Consider

$$\mathscr{H}^{ps} = \{(\bar\partial_E, \Phi) \in \mathscr{H} \quad : \quad (E, \Phi) \quad \text{is polystable}\},$$

  We define he **moduli space of Higgs bundle structures** supported by $\mathbb{E}$ as

$$\mathscr{H}^{ps}/\mathscr{G}^c.$$

Moduli spaces of Higgs bundles over Riemann surfaces

- Let h be a Hermitian metric on $\mathbb{E}$, and let $\mathscr{X} = \mathscr{A} \times \Omega^{1,0}(X, \operatorname{End}\mathbb{E})$.
- **Hitchin's equations** can be regarded as the system of equations for a pair $(d_A, \Phi) \in \mathscr{X}$ given by

$$\begin{aligned} &F_A + [\Phi, \Phi^*] = -i\mu I_{\mathbb{E}}\omega \\ &\bar{\partial}_A \Phi = 0, \end{aligned} \tag{3}$$

where $\bar{\partial}_A = d_A^{0,1}$ is the holomorphic structure defined by d_A
- The set $\mathscr{X}_0 = \{(d_A, \Phi) \in \mathscr{X}$ satisfying (3)$\}$ is invariant under the action of $\mathscr{G}$, and the **moduli space of solutions to Hitchin's equations** is defined as $\mathscr{X}_0/\mathscr{G}$.

Theorem (Hitchin, 1987; Simpson, 1988)

The correspondence $\mathscr{A} \times \Omega^{1,0}(X, \operatorname{End}\mathbb{E}) \leftrightarrow \mathscr{C} \times \Omega^{1,0}(X, \operatorname{End}\mathbb{E})$ induces a **bijection**

$$\mathscr{X}_0/\mathscr{G} \longleftrightarrow \mathscr{H}^{ps}/\mathscr{G}^c.$$

Some references

- M.F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, *Phil. Trans. R. Soc. Lond. A* **308** (1982), 523–615.
- S. K. Donaldson, A new proof a theorem of Narasimhan and Seshadri, *J. Diff. Geom.* **18**, (1982), 269–278.
- N.J. Hitchin, The self-duality equations on a Riemann surface, *Proc. Lond. Math. Soc.* **55** (1987), 59–126.
- S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Princeton University Press, New Jersey, 1987.
- M.S. Narasimhan and C.S. Seshadri, Stable and unitary bundles on a compact Riemann surface, *Ann. of Math.* **82** (1965), 540–564.
- C.T. Simpson, Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization, *J. Amer. Math. Soc.* **1** (1988), 867–918.
- R.O. Wells, *Differential Analysis on Complex Manifolds*, Springer, GTM 65, 3rd edition, 2007 (with an appendix by O. García-Prada).