# MOMENT MAP EQUATIONS IN GAUGE THEORY AND COMPLEX GEOMETRY

## Lecture 2

## Representations of the fundamental group and non-abelian Hodge correspondence

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# Connections and representations of the fundamental group

- Let $\mathbb{E}$ be a smooth complex vector bundle of rank $n$ over a compact Riemann surface $X$, and let $D$ be a connection on $\mathbb{E}$.
- A **section** $s \in \Omega^0(X, \mathbb{E})$ is said to be **parallel** if $Ds = 0$. If $\gamma = \gamma(t)$, $0 \leq t \leq T$ is a curve in $X$, a section $s$ defined along $\gamma$ is said to be **parallel along** $\gamma$ if

$$Ds(\gamma'(t)) = 0 \quad \text{for} \quad 0 \leq t \leq T, \tag{1}$$

  where $\gamma'(t)$ is the tangent vector of $\gamma$ at $\gamma(t)$.
- If $s_0$ is an element of the initial fibre $\mathbb{E}_{\gamma(0)}$, by solving the **system of ordinary differential equations** (1) with initial condition $s_0$ we can extend $s_0$ uniquelly to a parallel section $s$ along $\gamma$, called the **parallel displacement of** $s_0$ **along** $\gamma$.
- If the initial and the end points of $\gamma$ coincide so that $x_0 = \gamma(0) = \gamma(T)$ then the parallel displacement along $\gamma$ defines a **linear transformation** of the fibre $\mathbb{E}_{x_0} \cong \mathbb{C}^n$.

- We thus have a map

$$\{\text{closed paths based at } x_0\} \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

  whose image is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ called the **holonomy group** of $D$ at $x_0$.
  **Exercise 4**. If $D$ is **flat** the parallel displacement depends only on the **homotopy class** of the closed path and hence the holonomy map defines a **representation**

$$\rho : \pi_1(X, x_0) \longrightarrow \mathrm{GL}(n, \mathbb{C}).$$

- Conversely, given a representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}(n, \mathbb{C})$, one has a rank $n$ vector bundle given by $\mathbb{E} := \tilde{X} \times_\rho \mathbb{C}^n$, where $\tilde{X}$ is the **universal cover** of $X$, and $\tilde{X} \times_\rho \mathbb{C}^n$ is the quotient of $\tilde{X} \times \mathbb{C}^n$ by the action of $\pi_1(X, x_0)$. The **trivial connection** on $\tilde{X} \times \mathbb{C}^n$ descends to give a **flat connection** on $\mathbb{E}$, whose holonomy is the image of $\rho$.

# Theorem of Narasimhan–Seshadri revisited

- As we know, the existence of flat connections on $\mathbb{E}$ implies that the first Chern class of $\mathbb{E}$ must vanish. We will consider for now only this case.
- Let $\mathbb{E}$ be a smooth complex vector bundle of rank $n$ and $c_1(\mathbb{E}) = 0$ over $X$, and let $h$ be a Hermitian metric on $\mathbb{E}$. It is clear that the holonomy group of a unitary connection is a subgroup of the unitary group $\mathrm{U}(n)$.
- Let $\mathscr{A}_0 \subset \mathscr{A}$ be the set of flat $h$-unitary connections on $\mathbb{E}$. The holonomy map induces a **bijection**

$$\mathscr{A}_0/\mathscr{G} \longleftrightarrow \mathrm{Hom}(\pi_1(X), \mathrm{U}(n))/\,\mathrm{U}(n)$$

- We obtain then the **original formulation of the Narasimhan–Seshadri theorem**: There is a bijection (in fact a homeomorphism)

$$\mathrm{Hom}(\pi_1(X), \mathrm{U}(n))/\,\mathrm{U}(n) \longleftrightarrow \mathscr{C}^{ps}/\mathscr{G}^c.$$

# Harmonic metrics on flat bundles

- Relating Higgs bundles to representation of the fundamental group requires an extra important theorem.
- Let $\mathbb{E}$ be a smooth complex vector bundle of rank $n$ and $c_1(\mathbb{E}) = 0$ over $X$, and let $D$ be a **flat connection** on $\mathbb{E}$. Let $h$ be a **Hermitian metric** on $\mathbb{E}$. The decomposition

  $$\operatorname{End}\mathbb{E} = \operatorname{End}(\mathbb{E}, h) \oplus i \operatorname{End}(\mathbb{E}, h)$$

  allows us to write in a **unique** way

  $$D = d_A + \Psi,$$

  where $d_A$ is a $h$-unitary connection on $\mathbb{E}$ and $\Psi$ is a **1-form** with values in the bundle of skew-Hermitian endomorphisms of $\mathbb{E}$.
- The metric $h$ is said to be **harmonic** if

  $$d_A^* \Psi = 0, \qquad (2)$$

  where we use the **conformal (complex) structure** of $X$ to define $d_A^*$.

# Harmonic metrics on flat bundles

- To explain why the word **"harmonic"** is used here, recall that a Hermitian metric $h$ on $\mathbb{E}$ is simply a **section of the** $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$**-bundle** over $X$ naturally associated to $\mathbb{E}$. This can be viewed as a $\pi_1(X)$**-equivariant map**

$$\tilde{h} : \tilde{X} \longrightarrow \mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n),$$

  where $\tilde{X}$ is the universal cover of $X$.
- $d_A^*\Psi = 0$ is equivalent to the condition that the map $\tilde{h}$ be **harmonic**, in the sense that it **minimizes the energy** $\mathcal{E}(\tilde{h}) = \int_{\tilde{X}} |d\tilde{h}|^2$.
- In fact, the one-form $\Psi$ can be identified with the differential of $\tilde{h}$, and $d_A$ with the pull-back of the **Levi–Civita connection** on $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$.

# Harmonic metrics on flat bundles

- The flat connection $D$ is called **reductive** if the corresponding holonomy representation $\rho : \pi_1(X) \to \mathrm{GL}(n, \mathbb{C})$ is **completely reducible**.

**Theorem (Donaldson, 1987; Corlette, 1988)**

Let $D$ be a flat connection on $\mathbb{E}$. Then $\mathbb{E}$ admits a harmonic metric if and only if $D$ is reductive.

**Remarks**

- Donaldson proves this for $n = 2$. Corlette's proof extends to the case in which we replace $X$ by a **compact Riemannian manifold of arbitrary dimension**.
- Very recently **D. Wu and X. Zhang** (2023) have extended Corlette's result to arbitrary connections, **not necessarily flat**. Here, a **reductive connection** $D$ is defined as one for which any $D$-invariant subbundle has a $D$-invariant complement.

# Harmonic metrics on flat bundles

- As in the previous existence theorems, we can formulate Donaldson–Corlette's theorem as a correspondence between moduli spaces. To do that, we fix a Hermitian metric $h$ on $\mathbb{E}$.
- As we have seem, there is a bijection

$$\begin{array}{ccc} \mathscr{D} & \longrightarrow & \mathscr{A} \times \Omega^1(X, \mathrm{End}(\mathbb{E}, h)) \\ D & \mapsto & (d_A, \Psi), \end{array}$$

- Now, flatness of $D = d_A + \Psi$ and harmonicity combined are equivalent to the **flat harmonicity equations**

$$\begin{aligned} & F_A + \tfrac{1}{2}[\Psi, \Psi] = 0 \\ & d_A \Psi = 0 \\ & d_A^* \Psi = 0. \end{aligned} \tag{3}$$

# Harmonic metrics on flat bundles

- Let $\mathscr{Y} = \mathscr{A} \times \Omega^1(X, \mathrm{End}(\mathbb{E}, h))$ and let

$$\mathscr{Y}_0 = \{(d_A, \Psi) \in \mathscr{Y} \quad \text{satisfying} \quad (3)\}.$$

  The gauge group $\mathscr{G}$ acts on $\mathscr{Y}_0$, and $\mathscr{Y}_0/\mathscr{G}$ is the **moduli space of solutions to the flat harmonicity equations** (3).

- Let $\mathscr{D}_0 \subset \mathscr{D}$ the set of **flat connections** on $\mathbb{E}$, and $\mathscr{D}_0^+ \subset \mathscr{D}_0$ be the subset of **reductive flat connections** on $\mathbb{E}$. This is invariant under the action of $\mathscr{G}^c$.

**Theorem (Donaldson, 1987; Corlette, 1988)**

The bijection $\mathscr{D} \longleftrightarrow \mathscr{A} \times \Omega^1(X, \mathrm{End}(\mathbb{E}, h))$ induces a **one-to-one correspondence**

$$\mathscr{D}_0^+/\mathscr{G}^c \longleftrightarrow \mathscr{Y}_0/\mathscr{G},$$

which restricts to a bijection between the corresponding irreducible objects.

- To recall the description of the moduli space of Higgs bundles in **differential-geometric** terms, let $\mathbb{E}$ be a smooth complex vector bundle over $X$, and consider the **set of pairs**

$$\mathscr{H} = \{(\bar\partial_E, \Phi) \in \mathscr{C} \times \Omega^{1,0}(X, \operatorname{End}\mathbb{E}) \quad : \quad \bar\partial_E \Phi = 0\}.$$

- The gauge group $\mathscr{G}^c$ acts on $\Omega^{1,0}(X, \operatorname{End}\mathbb{E})$ by

$$g \cdot \Phi = g\Phi g^{-1} \quad \text{where} \quad g \in \mathscr{G}^c \quad \text{and} \quad \Phi \in \Omega^{1,0}(X, \operatorname{End}\mathbb{E}),$$

  This, combined with the action on $\mathscr{C}$, gives an action on $\mathscr{H}$.
- Consider

$$\mathscr{H}^{ps} = \{(\bar\partial_E, \Phi) \in \mathscr{H} \quad : \quad (E, \Phi) \quad \text{is polystable}\},$$

  We define he **moduli space of Higgs bundle structures** supported by $\mathbb{E}$ as

$$\mathscr{H}^{ps}/\mathscr{G}^c.$$

# Hitchin–Simpson correspondence revisited

- Let $h$ be a Hermitian metric on $\mathbb{E}$, and let $\mathscr{X} = \mathscr{A} \times \Omega^{1,0}(X, \operatorname{End}\mathbb{E})$.
- **Hitchin's equation** can be regarded as the system of equations for a pair $(d_A, \Phi) \in \mathscr{X}$ given by

$$\begin{aligned} F_A + [\Phi, \Phi^*] &= -i\mu I_{\mathbb{E}}\omega \\ \bar\partial_A \Phi &= 0, \end{aligned} \tag{4}$$

  where $\bar\partial_A = d_A^{0,1}$ is the holomorphic structure defined by $d_A$
- The set $\mathscr{X}_0 = \{(d_A, \Phi) \in \mathscr{X}$ satisfying (4)$\}$ is invariant under the action of $\mathscr{G}$, and the **moduli space of solutions to Hitchin's equations** is defined as $\mathscr{X}_0/\mathscr{G}$.

**Theorem (Hitchin, 1987; Simpson, 1988)**

The correspondence $\mathscr{A} \times \Omega^{1,0}(X, \operatorname{End}\mathbb{E}) \leftrightarrow \mathscr{C} \times \Omega^{1,0}(X, \operatorname{End}\mathbb{E})$ induces a **bijection**

$$\mathscr{X}_0/\mathscr{G} \longleftrightarrow \mathscr{H}^{ps}/\mathscr{G}^c.$$

- A Hermitian metric $h$ on $\mathbb{E}$ defines a bijection

$$\begin{array}{ccc} \Omega^{1,0}(X, \operatorname{End}\mathbb{E}) & \longrightarrow & \Omega^1(X, \operatorname{End}(\mathbb{E}, h)) \\ \Phi & \mapsto & \Psi = \Phi + \Phi^*. \end{array}$$

  We have the following.

- The pair $(d_A, \Psi)$ satisfies the harmonicity equations (3) if and only if $(d_A, \Phi)$ satisfies Hitchin's equations (4), where $\Psi = \Phi + \Phi^*$.

**Hitchin equations $\Longleftrightarrow$ Harmonicity equations**

We have a bijection

$$\mathscr{X}_0/\mathscr{G} \longleftrightarrow \mathscr{Y}_0/\mathscr{G},$$

which restricts to a homeomorphism between the moduli spaces of irreducible solutions.

# Non-abelian Hodge correspondence

- Combining this with Hitchin–Simpson correspondence and Donaldson–Corlette correspondence we obtain the following.

**Non-abelian Hodge correspondence**

There is a one-to-one correspondence

$$\mathscr{D}_0^+/\mathscr{G}^c \longleftrightarrow \mathscr{H}^{ps}/\mathscr{G}^c.$$

- Let $\mathcal{R}(n) = \mathrm{Hom}^+(\pi_1(X), \mathrm{GL}(n, C))/\,\mathrm{GL}(n, \mathbb{C})$ be the **moduli space of reductive representations** of $\pi_1(X)$ in $\mathrm{GL}(n, \mathbb{C})$. Of course $\mathcal{R}_n$ is in bijection with $\mathscr{D}_0^+/\mathscr{G}^c$.
- Let $\mathcal{M}(n)$ the **moduli space of polystable Higgs bundles** of rank $n$ and degree 0. We have the following.

**Non-abelian Hodge correspondence**

There is a **homeomorphism**

$$\mathcal{R}(n) \cong \mathcal{M}(n).$$

# Non-abelian Hodge correspondence

- The Non-abelian Hodge correspondence can be extended to Higgs bundles of **arbitrary degree**.
- If $c_1(\mathbb{E}) \neq 0$, we can consider connections with central curvature. These connections on $\mathbb{E}$ induce flat connections on the **principal** $\mathrm{PGL}(n,\mathbb{C})$**-bundle** associated to $\mathbb{E}$. The **holonomy map of a projectively flat connection** defines a homomorphism

$$\tilde{\rho} : \pi_1(X) \longrightarrow \mathrm{PGL}(n,\mathbb{C}) = \mathrm{GL}(n,\mathbb{C})/\mathbb{C}^*.$$

- $\pi_1(X)$ is generated by $2g$ generators ($g$ is the genus), say $A_1, B_1, \ldots, A_g, B_g$, subject to the single relation $\prod_{i=1}^{g}[A_i, B_i] = 1$, and has a **universal central extension**

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1(X) \longrightarrow 1$$

generated by the same generators as $\pi_1(X)$, together with a **central element** $J$ subject to the **relation** $\prod_{i=1}^{g}[A_i, B_i] = J$.

- By the **universal property** of $\Gamma$, we **can lift** every $\tilde{\rho}:\pi_1(X)\longrightarrow \mathrm{PGL}(n,\mathbb{C})$ to a representation $\rho$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma & \longrightarrow & \pi_1(X) & \longrightarrow & 1 \\ & & \downarrow & & \rho\downarrow & & \tilde{\rho}\downarrow & & \\ 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{GL}(n,\mathbb{C}) & \longrightarrow & \mathrm{PGL}(n,\mathbb{C}) & \longrightarrow & 1. \end{array}$$

- To a central representation $\rho:\Gamma\to \mathrm{GL}(n,\mathbb{C})$ we can associate a **topological invariant** given by the degree $\deg(E_\rho)\in\mathbb{Z}$ of the vector bundle $E_\rho$ with corresponding central connection.

# Non-abelian Hodge correspondence

- Let $\mathcal{R}(n,d)$ be the moduli space of **reductive central representations** of $\Gamma$ in $\mathrm{GL}(n,\mathbb{C})$, and $\mathcal{M}(n,d)$ be the **moduli space of polystable Higgs bundles** of rank $n$ and degree $d$. Then

**Non-abelian Hodge correspondence**

There is a **homeomorphism**

$$\mathcal{R}(n,d) \cong \mathcal{M}(n,d).$$

- If we consider the moduli space $R(n,d)$ of central representations of $\Gamma$ in $\mathrm{U}(n)$ of degree $d$, and $M(n,d)$ is the **moduli space of polystable vector bundles** of rank $n$ and degree $d$. Of course, we have the following.

**Narasimhan–Seshadri theorem**

There is a **homeomorphism**

$$R(n,d) \cong M(n,d).$$

- **Question**: How about representations of $\pi_1(X)$ or $\Gamma$ in **non-compact real forms** of $\mathrm{GL}(n,\mathbb{C})$? Is there a similar kind of correspondence? The answer is **yes** — we will ilustrate this with the real forms $\mathrm{U}(p,q)$.
- The group $\mathrm{U}(p,q)$, with $p+q=n$, is defined as the group of linear transformations of $\mathbb{C}^n$ which preserve the **Hermitian metric of signature** $(p,q)$ defined by

$$\langle z,w\rangle = z_1\overline{w}_1+\cdots+z_p\overline{w}_p-\cdots-z_{p+1}\overline{w}_{p+1}-z_{p+q}\overline{w}_{p+q},$$

for $z=(z_1,\cdots,z_n)\in\mathbb{C}^n$ and $w=(w_1,\cdots,w_n)\in\mathbb{C}^n$. If

$$I_{p,q}=\begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix},$$

we have that

$$\mathrm{U}(p,q)=\{A\in\mathrm{GL}(n,\mathbb{C}) \;\;:\;\; AI_{p,q}\overline{A}^t=I_{p,q}\}.$$

# Representations in $\mathrm{U}(p,q)$ and Higgs bundles

- We need to consider a smooth complex vector bundle $\mathbb{E}$ equipped with a $\mathrm{U}(p,q)$-**structure** $H$, i.e. a Hermitian metric of signature $(p,q)$.
- Such a bundle has a **finer topological invariant** than its **degree** $d$: We first observe that $\mathrm{U}(p)\times\mathrm{U}(q)\subset\mathrm{U}(p,q)$ is a **maximal compact subgroup** of $\mathrm{U}(p,q)$. We can **reduce the structure group** of $(E,H)$ to the group $\mathrm{U}(p)\times\mathrm{U}(q)$, and hence $\mathbb{E}\cong\mathbb{V}\oplus\mathbb{W}$, where $\mathbb{V}$ and $\mathbb{W}$ are vector bundles with $\operatorname{rank}\mathbb{V}=p$ and $\operatorname{rank}\mathbb{W}=q$, naturally equipped with **Hermitian metrics** $h_{\mathbb{V}}$ and $h_{\mathbb{W}}$, respectively.
- The **topological invariant** naturally associated to $(\mathbb{E},H)$ is the **pair of integers** $(a,b)$, where $a=\deg\mathbb{V}$ and $b=\deg\mathbb{W}$, with $d=a+b$.

# Representations in $\mathrm{U}(p,q)$ and Higgs bundles

- Let $\rho \in \mathrm{Hom}(\Gamma, \mathrm{U}(p,q))$ be a central representation of $\Gamma$ in $\mathrm{U}(p,q)$. As in the case of $\mathrm{U}(n)$ and $\mathrm{GL}(n,\mathbb{C})$, to $\rho$ we can associate a smooth vector bundle $E_\rho$ equipped with a $\mathrm{U}(p,q)$ structure and a $\mathrm{U}(p,q)$-connection with central curvature.
- Consider the **moduli space** $\mathcal{R}(p,q,a,b)$ **of reductive central representations** of $\Gamma$ in $\mathrm{U}(p,q)$ with invariant $(a,b) \in \mathbb{Z} \times \mathbb{Z}$. Of course the representations for which $a+b=0$ correspond to representations of the **fundamental group** of $X$.
- Given a representation of $\Gamma$ in $\mathrm{U}(p,q)$ with topological invariant $c(\rho) = (a,b)$, the **Toledo invariant** of $\rho$ is defined by

$$\tau(\rho) = \tau(p,q,a,b) = 2\frac{qa-pb}{p+q}.$$

**Theorem: Milnor–Wood inequality (Domic–Toledo, 1987)**

$$|\tau(p,q,a,b)| \leq \min\{p,q\}(2g-2).$$

- There is a special class of Higgs bundles called $\mathrm{U}(p,q)$-**Higgs bundles**, given by

$$\left(E = V \oplus W, \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}\right),$$

where $V$ and $W$ are holomorphic vector bundles of rank $p$ and $q$ respectively and the non-zero components in the Higgs field are $\beta \in H^0(\mathrm{Hom}(W,V)\otimes K)$, and $\gamma \in H^0(\mathrm{Hom}(V,W)\otimes K)$.
- Let $(a,b) \in \mathbb{Z}\times\mathbb{Z}$. Define the **moduli space of polystable** $\mathrm{U}(p,q)$-**Higgs bundles** $\mathcal{M}(p,q,a,b)$ as the set of isomorphism classes of polystable $\mathrm{U}(p,q)$-Higgs bundles with $\deg(V)=a$ and $\deg W = b$.

**Theorem (Bradlow–G–Gothen, 2003)**

There is a homeomorphism

$$\mathcal{M}(p,q,a,b) \cong \mathcal{R}(p,q,a,b),$$

# Representations in $\mathrm{U}(p,q)$ and Higgs bundles

- The Milnor–Wood inequality for the Toledo invariant can be derived from the **polystability of the corresponding Higgs bundle** $(E,\Phi)\in\mathcal{M}(p,q,a,b)$.

**Theorem (Bradlow–G–Gothen, 2003, Bradlow–G–Gothen–Heinloth, 2018)**

The moduli space $\mathcal{M}(p,q,a,b)$ (and hence $\mathcal{R}(p,q,a,b)$) is a **non-empty** and **connected** if and only if $|\tau(p,q,a,b)|\leq\min\{p,q\}(2g-2)$.

**Theorem: Rigidity (Bradlow–G–Gothen, 2003)**

Let $p\neq q$. If $|\tau|=\min\{p,q\}(2g-2)$, then $\mathcal{M}(p,q,a,b)$ consists entirely of polystable (non-stable) objects, and hence has **smaller dimension** than expected.

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