MOMENT MAP EQUATIONS IN GAUGE THEORY AND COMPLEX GEOMETRY
Lecture 3
Moment maps and moduli spaces

Oscar García-Prada
ICMAT-CSIC, Madrid

Cargèse, 27 June 2024

Symplectic and Kähler quotients

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  A **Kähler manifold** with its Kähler form is an example of a symplectic manifold.
- A transformation f of M is called **symplectic** if it leaves invariant the 2-form , i.e., $f^*\omega = \omega$.
- Let G be **Lie group acting symplectically** on (M, ω). If v is a **vector field** generated by the action, then $L_v\omega = 0$. Since $L_v\omega = i(v)d\omega + d(i(v)\omega)$, hence $d(i(v)\omega) = 0$. If there exists a function $\mu_v : M \to \mathbb{R}$ such that

$$d\mu_v = i(v)\omega.$$

  the function μ_v is said to be a **Hamiltonian function** for the vector field v.

Symplectic and Kähler quotients

- As v ranges over the set of vector fields generated by the elements of the Lie algebra $\mathfrak{g}$ of G, these functions can be chosen to fit together to give a map

$$\mu : M \longrightarrow \mathfrak{g}^*,$$

defined by

$$\langle \mu(x), a \rangle = \mu_{\tilde{a}}(x),$$

where $\tilde{a}$ is the **vector field generated** by $a \in \mathfrak{g}$, $x \in X$ and $\langle \cdot, \cdot \rangle$ is the **natural pairing** between $\mathfrak{g}$ and its dual.

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- There is a natural action of G on both sides and a constant ambiguity in the choice of μ_v. If this can be adjusted so that μ is G-**equivariant**, i.e.

$$\mu(g(x)) = (\mathrm{Ad}\, g)^*(\mu(x)) \quad \text{for} \quad g \in G \quad x \in M,$$

then μ is called a **moment map** for the action of G on M.

Symplectic and Kähler quotients

- Moment maps give a way of **constructing new symplectic manifolds**. More precisely, suppose that G acts **freely and discontinuously** on $\mu^{-1}(0)$ (recall that $\mu^{-1}(0)$ is G-invariant), then
$$\mu^{-1}(0)/G$$
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- This **symplectic reduction** process is valid for **infinite dimensional Banach manifolds** acted upon by **infinite dimensional Banach Lie groups**.

Symplectic and Kähler quotients

- Suppose now that M has a **Kähler structure**. It is convenient to describe a Kähler structure on the manifold M as a triple (g, J, ω) consisting of a **Riemannian metric** g, an **integrable almost complex structure** (a complex structure) J and a **symplectic form** ω on M which satisfies

$$\omega(u, v) = g(Ju, v), \quad \text{for } x \in M \text{ and } u, v \in T_x M.$$

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  Any two of these structures determines the third one.

- Let G now be a Lie group acting on (M, g, J, ω) preserving the Kähler structure. Then if $\mu : M \longrightarrow \mathfrak{g}^*$ is a **moment map**, and G **acts freely and discontinuously** on $\mu^{-1}(\lambda)$, for a central element $\lambda \in \mathfrak{g}^*$, the quotient $\mu^{-1}(\lambda)/G$ is also a **Kähler manifold**. This process is called **Kähler reduction**.

GIT quotients

- When M is a **projective algebraic manifold** there is a very important relation between the symplectic quotient and the algebraic quotient defined by **Mumford's Geometric Invariant Theory (GIT)**.

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- $x \in M$ is **semistable** if there is a non-constant invariant polynomial f with $f(x) \neq 0$. This is equivalent to saying that if $\tilde{x} \in \mathbb{C}^n$ is any representative of x, then **the closure of the** G^c**-orbit of** $\tilde{x}$ **does not contain the origin**. Let $M^{ss} \subset M$ the set of all semistable points.

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- There is a subset $M^s \subset M^{ss}$ of **stable** points which satisfy the stronger condition that **the** G^c**-orbit of** $\tilde{x}$ **is closed in** $\mathbb{C}^n$.

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Theorem (Mumford, Kempf–Ness, Guillemin and Sternberg...)

$$\mu^{-1}(0)/G \cong M /\!\!/ G^c.$$

Connections and moment maps

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- The set $\mathscr{A}$ of h-**unitary connections** on $\mathbb{E}$ is an **affine space** modelled on $\Omega^1(X, \mathrm{End}(\mathbb{E}, h))$, and is equipped with a **symplectic structure** defined by

$$\omega_{\mathscr{A}}(\psi, \eta) = \int_X \mathrm{Tr}(\psi \wedge \eta), \quad A \in \mathscr{A}, \quad \psi, \eta \in T_A\mathscr{A} = \Omega^1(\mathrm{End}(\mathbb{E}, h)).$$

This is **obviously closed** since it is independent of $A \in \mathscr{A}$.

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  This is **obviously closed** since it is independent of $A \in \mathscr{A}$.
- The set $\mathscr{C}$ of **holomorphic structures** on $\mathbb{E}$ is an **affine space modelled** on $\Omega^{0,1}(X, \mathrm{End}\,\mathbb{E})$, and has a **complex structure** $J_{\mathscr{C}}$, induced by the complex structure of the Riemann surface, which is defined by

$$J_{\mathscr{C}}(\alpha) = i\alpha, \quad \text{for } \bar\partial_E \in \mathscr{C} \text{ and } \alpha \in T_{\bar\partial_E}\mathscr{C} = \Omega^{0,1}(X, \mathrm{End}\,\mathbb{E}).$$

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- We first observe that $\mathrm{Lie}\,\mathscr{G} = \Omega^0(X, \mathrm{End}(\mathbb{E}, h))$ is **canonically dual** to $\Omega^2(X, \mathrm{End}(\mathbb{E}, h))$, i.e., $\mathrm{Lie}\,\mathscr{G}^* = \Omega^2(X, \mathrm{End}(\mathbb{E}, h))$. More concretely, let $a \in \Omega^0(X, \mathrm{End}(\mathbb{E}, h))$ and $\alpha \in \Omega^2(X, \mathrm{End}(\mathbb{E}, h))$:

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Theorem (Atiyah–Bott, 1982)

There is a **moment map** for the action of $\mathscr{G}$ on $\mathscr{A}$ given by

$$\begin{array}{ccc} \mathscr{A} & \longrightarrow & \Omega^2(X, \mathrm{End}(\mathbb{E}, h)) \\ A & \longmapsto & F_A. \end{array}$$

- **To prove this**, let $a \in \operatorname{Lie} \mathscr{G} = \Omega^0(X, \operatorname{End}(\mathbb{E}, h))$, and let $\tilde{a}$ be the vector field generated by a. We have to show that the function $\mu_{\tilde{a}} : \mathscr{A} \to \mathbb{R}$ given by

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  is **Hamiltonian**.
- Equivalently we have to show that

$$d\mu_{\tilde{a}}(A)(v) = \omega_{\mathscr{A}}(v, \tilde{a}) = \int_X \mathrm{Tr}(v \wedge \tilde{a})$$

- **Exercise 5**: This follows from:
  1. $\tilde{a} = d_A a$,
  2. $$d\mu_{\tilde{a}}(A)(v) = \int_X \mathrm{Tr}(a \wedge d_A v),$$
  3. $$\int_X \mathrm{Tr}(a \wedge d_A v) = -\int_X \mathrm{Tr}(d_A a \wedge v).$$

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- In order to have a **non-empty** symplectic reduction, we take the central element $\lambda \in \Omega^2(X, \mathrm{End}(\mathbb{E}, h))$ given by $\lambda = -i\mu I_{\mathbb{E}}\omega$, and consider $\mu^{-1}(\lambda)$.

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- This coincides with the set

$$\mathscr{A}_0 := \{A \in \mathscr{A} \quad : \quad F_A = -i\mu I_{\mathbb{E}}\omega\}$$

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- In view of this, the correspondence given by the Narasimhan–Seshadri Theorem

$$\mu^{-1}(\lambda)/\mathscr{G} \longleftrightarrow \mathscr{C}^{ps}/\mathscr{G}^c$$

  is formally an **infinite dimensional version** of the isomorphism between the symplectic and the algebraic quotients in finite dimensions.

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- In general, the space $\mu^{-1}(\lambda)/\mathscr{G}$ has singularities, but if we restric μ to the open subspace in $\mathscr{A}_0$ of **irreducible connections** then $\mu^{-1}(\lambda)/\mathscr{G}$ is trully a smooth Kähler manifold, which is identified by the NS theorem with the moduli space of **stable** vector bundles.

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- Note that even though $\mathscr{A}$ is infinite dimensional, the symplectic reduction obtained has **finite dimension**. The central curvature condition and the action of the gauge group defined a **deformation complex** which is **elliptic**.

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- Let us denote $\Omega = \Omega^{1,0}(X, \operatorname{End} \mathbb{E})$. The linear space Ω has a natural complex structure J_Ω defined by multiplication by i, and a symplectic structure given by

$$\omega_\Omega(\psi, \eta) = i \int_X \operatorname{Tr}(\psi \wedge \eta^*), \quad \text{for } \Phi \in \Omega \text{ and } \psi, \eta \in T_\Phi \Omega = \Omega.$$

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- We can now consider $\mathscr{X} = \mathscr{A} \times \Omega$ with the symplectic structure $\omega_{\mathscr{X}} = \omega_{\mathscr{A}} + \omega_\Omega$ and complex structure $J_{\mathscr{X}} = J_{\mathscr{A}} + J_\Omega$.

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- The action of $\mathscr{G}$ on $\mathscr{X}$ preserves $\omega_{\mathscr{X}}$ and $J_{\mathscr{X}}$ and there is a moment map

$$\begin{array}{cccc} \mu_{\mathscr{X}}: & \mathscr{X} & \longrightarrow & \Omega^2(X, \mathrm{End}(\mathbb{E}, h)) \\ & (A, \Phi) & \mapsto & F_A + [\Phi, \Phi^*]. \end{array}$$

Higgs bundles and moment maps

- We now consider the **subvariety** of $\mathscr{X} = \mathscr{A} \times \Omega$

$$\mathscr{N} = \{(d_A, \Phi) \in \mathscr{X} \quad : \quad \bar{\partial}_A \Phi = 0\},$$

corresponding to the space

$$\mathscr{H} = \{(\bar{\partial}_E, \Phi) \in \mathscr{C} \times \Omega^{1,0}(X, \operatorname{End} \mathbb{E}) \quad : \quad \bar{\partial}_E \Phi = 0\}.$$

under the identification between $\mathscr{A}$ and $\mathscr{C}$ given by the Chern correspondence. Avoiding difficulties with possible singularities, $\mathscr{N}$ inherits a Kähler structure from $\mathscr{X}$.

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- Since $\mathscr{N}$ is $\mathscr{G}$-invariant, the moment map is the restriction

$$\mu = \mu_{\mathscr{X}}|_{\mathscr{N}} : \mathscr{N} \longrightarrow \Omega^2(X, \operatorname{End}(\mathbb{E}, h)).$$

Now the **Kähler quotient**

$$\mu^{-1}(\lambda)/\mathscr{G}.$$

is the **moduli space of solutions to Hitchin equations**.

- Consider Hermitian bundles $(\mathbb{V}, h_{\mathbb{V}})$ and $(\mathbb{W}, h_{\mathbb{W}})$ of rank p and q respectively and let $\mathscr{A}_{\mathbb{V}}$ and $\mathscr{A}_{\mathbb{W}}$ be the corresponding spaces of unitary connections.

U(p, q)-Higgs bundles and moment maps

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- The space $\mathscr{Y}$ is a **Kähler submanifold** of $\mathscr{A} \times \Omega$ which is invariant and the subgroup $\mathscr{G}_{\mathbb{V}} \times \mathscr{G}_{\mathbb{W}} \subset \mathscr{G}$.

U(p, q)-Higgs bundles and moment maps

- The moment map is hence given by **projecting onto**

$$\Omega^2(X, \mathrm{End}(\mathbb{V}, h_{\mathbb{V}})) \oplus \Omega^2(X, \mathrm{End}(\mathbb{W}, h_{\mathbb{W}}))$$

sending

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- We can then restrict this to obtain a moment map μ on the $(\mathscr{G}_{\mathbb{V}} \times \mathscr{G}_{\mathbb{W}})$-invariant Kähler submanifold $\mathscr{N}_{\mathscr{Y}} = \mathscr{N} \cap \mathscr{Y}$, where $\mathscr{N}$ is given above.

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- The quotient $\mu^{-1}(\lambda)/\mathscr{G}_{\mathbb{V}} \times \mathscr{G}_{\mathbb{W}}$ is the **moduli space of solutions to the $\mathrm{U}(p,q)$-Hitchin equations**, which is isomorphic to the **moduli space of $\mathrm{U}(p,q)$-Higgs bundles** $\mathcal{M}(p,q,a,b)$ where a and b are the Chern classes of $\mathbb{V}$ and $\mathbb{W}$ respectively.

Hyperkähler quotients

- A **hyperkähler manifold** is a differentiable manifold M equipped with a Riemannian metric g and **complex structures** J_i, $i = 1, 2, 3$ satisfying the **quaternion relations** $J_i^2 = -I$, $J_3 = J_1 J_2$, etc., such that if we define $\omega_i(\cdot, \cdot) = g(J_i \cdot, \cdot)$, (g, J_i, ω_i) is a **Kähler structure** on M.

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- Let G be a Lie group acting on M **preserving the Kähler structures** (g, J_i, ω_i) and having **moment maps** $\mu_i : X \to \mathfrak{g}^*$ for $i = 1, 2, 3$. We can combine these moment maps in a map

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- Let $\lambda_i \in \mathfrak{g}^*$ for $i = 1, 2, 3$ be central elements and consider the G-invariant submanifold $\boldsymbol{\mu}^{-1}(\boldsymbol{\lambda})$ where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. Then if G acts on $\boldsymbol{\mu}^{-1}(\boldsymbol{\lambda})$ freely and discontinuously the quotient

$$\boldsymbol{\mu}^{-1}(\boldsymbol{\lambda})/G$$

is a **hyperkähler manifold**.

- Let $(\mathbb{E}, h)$ be a smooth complex Hermitian vector bundle over a compact Riemann surface X, equipped with a Kähler form ω.

Hyperkähler quotients and Hitchin equations

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- Recall that $\mathscr{X} = \mathscr{A} \times \Omega$ has a **Kähler structure** defined by $J_{\mathscr{X}}$ and $\omega_{\mathscr{X}}$. Let us rename $J_1 = J_{\mathscr{X}}$.

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- Recall that $\mathscr{X} = \mathscr{A} \times \Omega$ has a **Kähler structure** defined by $J_{\mathscr{X}}$ and $\omega_{\mathscr{X}}$. Let us rename $J_1 = J_{\mathscr{X}}$.
- Via the identification $\mathscr{A} \cong \mathscr{C}$, we have for $\alpha \in \Omega^{0,1}(X, \operatorname{End} \mathbb{E})$ and $\psi \in \Omega^{1,0}(X, \operatorname{End} \mathbb{E})$ the following three complex structures on $\mathscr{X}$:

$$\begin{aligned} J_1(\alpha, \psi) &= (i\alpha, i\psi) \\ J_2(\alpha, \psi) &= (i\psi^*, -i\alpha^*) \\ J_3(\alpha, \psi) &= (-\psi^*, \alpha^*), \end{aligned}$$

where α^* and ψ^* is defined using the Hermitian metric h on $\mathbb{E}$.

- Clearly, J_i, $i = 1, 2, 3$ satisfy the quaternion relations, and define a **hyperkähler structure** on $\mathscr{X}$, with symplectic structures ω_i, $i = 1, 2, 3$, where $\omega_1 = \omega_{\mathscr{X}}$.

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- The action of $\mathscr{G}$ on $\mathscr{X}$ **preserves the hyperkähler structure** and there are **moment maps** given by

$$\mu_1(A,\Phi) = F_A + [\Phi, \Phi^*],\ \mu_2(A,\Phi) = \mathrm{Re}(\bar{\partial}_A\Phi),\ \mu_3(A,\Phi) = \mathrm{Im}(\bar{\partial}_A\Phi).$$

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- Taking $\boldsymbol{\lambda} = (\lambda, 0, 0)$, where $\lambda = -i\mu I_{\mathbb{E}}\omega$, $\boldsymbol{\mu}^{-1}(\boldsymbol{\lambda})/\mathscr{G}$ is the **moduli space of solutions to Hitchin equations**. In particular, if we consider the **irreducible solutions** $\boldsymbol{\mu}_*^{-1}(\boldsymbol{\lambda})$ we have that

$$\boldsymbol{\mu}_*^{-1}(\boldsymbol{\lambda})/\mathscr{G}$$

is a **hyperkähler manifold** which, by the **Hitchin–Simpson correspondence**, is isomorphic to the **moduli space** $\mathcal{M}^s(n, d)$ **of stable Higgs bundles** of rank n and d.

Hyperkähler quotients and flat harmonicity equations

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- Using the **complex structure** of X we have also the **complex structure** $I_2 = i \otimes \tau$, where $\tau(\psi) = \psi^*$ is the involution defined by the Hermitian metric h. We can finally consider the complex structure $I_3 = I_1 I_2$.

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- The Hermitian metric on $\mathbb{E}$ together with a Riemannian metric of X defines a **flat Riemannian metric** $g_{\mathscr{D}}$ on $\mathscr{D}$ which is kähler for the above three complex structures. Hence $(\mathscr{D}, g_{\mathscr{D}}, I_1, I_2, I_3)$ is also a **hyperkähler manifold**.

Hyperkähler quotients and harmonicity equations

- As in the previous case, the action of the gauge group $\mathscr{G}$ on $\mathscr{D}$ **preserves the hyperkähler structure** and there are moment maps

$$\mu_1(D) = d_A^*\Psi, \quad \mu_2(D) = \mathrm{Im}(F_D), \quad \mu_3(D) = \mathrm{Re}(F_D),$$

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- Hence the **moduli space of solutions to the flat harmonicity equations** is the **hyperkähler quotient** defined by

$$\boldsymbol{\mu}^{-1}(0, \lambda, 0)/\mathscr{G},$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ and $\lambda = -i\mu I_{\mathbb{E}}\omega$.

- The homeomorphism between the moduli spaces of solutions to the Hitchin and the flat harmonicity equations is induced from the **hypercomplex affine map**

$$\begin{array}{ccc} \mathscr{A} \times \Omega & \longrightarrow & \mathscr{D} \\ (d_A, \Phi) & \longmapsto & d_A + \Phi + \Phi^*. \end{array}$$

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- This identifies the hyperkähler quotient to the set of equivalence classes of polystable pairs on $\mathbb{E}$.
- If one now takes J_2 on $\mathscr{A} \times \Omega$ or **equivalently** $\mathscr{D}$ **with** I_1 and argues in a similar way, one gets the **Donaldson–Corlette**

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