MOMENT MAP EQUATIONS IN GAUGE THEORY AND COMPLEX GEOMETRY

Lecture 4

Kähler–Yang–Mills equations and gravitating vortices

Oscar García-Prada
ICMAT-CSIC, Madrid

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Kähler–Yang–Mills equations

- M compact complex manifold of $\dim_{\mathbb{C}} M = n$
- $E \to M$ holomorphic vector bundle over M

Kähler–Yang–Mills equations

for a **Kähler metric** g on M and a **Hermitian metric** h on E:

$$i\Lambda_g F_h = \lambda \operatorname{Id}_E$$
$$S_g - \alpha\Lambda_g^2 \operatorname{Tr} F_h^2 = c$$

- $F_h \in \Omega^2(M, \operatorname{End}(E,h))$ **curvature of Chern connection**
- Λ_g is contraction with the Kähler form
- S_g **scalar curvature** of g
- $\alpha > 0$ coupling constant
- $\lambda \in \mathbb{R}$ and $c \in \mathbb{R}$ determined by the topology

Kähler–Yang–Mills equations

- **Taking traces** in the first equation and integrating over M:

$$\lambda = \frac{2\pi}{\mathrm{vol}_g(M)} \frac{\deg_g(E)}{\mathrm{rank}\, E}$$

where

$$\deg_g(E) = \frac{1}{(n-1)!} \int_M c_1(E) \omega_g^{n-1}$$

- **Integrating** the second equation over M:

$$\begin{aligned} c\, \mathrm{vol}_g(M) &= \int_M (S_g - \alpha \Lambda_g^2 \operatorname{Tr} F_h^2) \\ &= \deg_g(TM) - \alpha \int_M ch_2(E) \wedge \omega_g^{n-2} \end{aligned}$$

Kähler–Yang–Mills equations

These equations were introduced in:

[AGG2013] L. Álvarez-Cónsul, M. García-Fernández and O. García-Prada, Coupled equations for Kähler metrics and Yang–Mills connections, *Geometry and Topology* **17** (2013) 2731–2812.

Based on:

Mario García-Fernández's PhD Thesis (2009).

Moment map interpretation of KYM equations

- Let M be a **smooth compact manifold** of real dimension $2n$ and E be **smooth complex vector bundle** over M.
- Fix: **symplectic form** ω on M & **Hermitian metric** h on E
- Consider the infinite-dimensional manifolds:

$$\mathscr{J} := \{\text{almost complex structures } J \text{ on } M \text{ compatible with } \omega\}$$
$$\mathscr{A} := \{\text{unitary connections on } (E,h)\}$$

- We have already seen that $\mathscr{A}$ is an infinite dimensional Kähler manifold.

Moment map interpretation of KYM equations

- The space $\mathscr{J}$ of **almost complex structures** is also an **infinite dimensional Kähler manifold**. The **complex structure** $\mathbf{J}\colon T_J\mathscr{J} \to T_J\mathscr{J}$ and **Kähler form** $\omega_{\mathscr{J}}$ are given by

$$\mathbf{J}\phi = J\phi$$

and

$$\omega_{\mathscr{J}}(\psi, \phi) = \frac{1}{2(n-1)!}\int_M \operatorname{Tr}(J\psi\phi)\omega^n,$$

for ϕ, $\psi \in T_J\mathscr{J}$, respectively. Here we identify $T_J\mathscr{J}$ with the space of endomorphisms $\phi\colon TM \to TM$ such that ϕ is symmetric with respect to the induced metric $\omega(\cdot, J\cdot)$ and satisfies $\phi J = -J\phi$.

Moment map interpretation of KYM equations

- Define

$$\mathscr{P} := \text{pairs } (J, A) \in \mathscr{J} \times \mathscr{A}$$

such that:

- (M, J, ω) is Kähler

- A induces a holomorphic structure on E over (M, J)

- Consider for fixed $\alpha \neq 0$ the **symplectic form** on $\mathscr{P}$:

$$\omega_\alpha := (\omega_{\mathscr{J}} + \alpha\omega_{\mathscr{A}})|_{\mathscr{P}}$$

- $\mathscr{P}$ is an **infinite dimensional Kähler submanifold** of $\mathscr{J} \times \mathscr{A}$.

Moment map interpretation of KYM equations

Group action?

Atiyah–Bott–Donaldson:

- $\mathscr{G}$ group of automorphisms of (E, h) covering the identity on M
- $\mathscr{G}$ acts symplectically on $(\mathscr{A}, \omega_{\mathscr{A}})$ with moment map $\mu_{\mathscr{A}} : \mathscr{A} \to (\mathrm{Lie}\,\mathscr{G})^*$ such that

$$\mu_{\mathscr{A}}(A) = 0 \Longleftrightarrow i\Lambda F_A = \lambda\,\mathrm{Id}_E$$

Fujiki–Donaldson–Quillen:

- $\mathscr{H} := \{\text{Hamiltonian symplectomorphisms } (M, \omega) \to (M, \omega)\}$
- $\mathscr{H}$ acts symplectically on $(\mathscr{J}, \omega_{\mathscr{J}})$ with moment map $\mu_{\mathscr{J}} : \mathscr{J} \to (\mathrm{Lie}\,\mathscr{H})^*$ such that

$$\mu_{\mathscr{J}}(J) = 0 \Longleftrightarrow S_{J,\omega} = \text{constant}$$

Hamiltonian extended gauge group $\widetilde{\mathscr{G}}$:
Automorphisms of (E, h) covering Hamiltonian symplectomorphims of (M, ω)

$$\begin{array}{ccc} (E,h) & \xrightarrow{g} & (E,h) \\ \downarrow & & \downarrow \\ (M,\omega) & \xrightarrow{\check{g}} & (M,\omega) \end{array}$$

Extension

$$1 \to \mathscr{G} \to \widetilde{\mathscr{G}} \to \mathscr{H} \to 1$$

$\mathscr{G}$: group of automorphisms of (E, h) covering the identity on M
$\mathscr{H}$: group of Hamiltonian symplectomorphisms of (M, ω)

Moment map interpretation of KYM equations

- $\widetilde{\mathscr{G}}$ acts on $\mathscr{J}$ via $\widetilde{\mathscr{G}} \to \mathscr{H}$: $g \mapsto \check{g}$
- $\widetilde{\mathscr{G}}$ acts on $\mathscr{A}$ in the usual way

Proposition

The action of $\widetilde{\mathscr{G}}$ on $(\mathscr{P}, \omega_\alpha)$ has moment map

$$\mu_\alpha : \mathscr{P} \to (\mathrm{Lie}\,\widetilde{\mathscr{G}})^*$$

such that

$$\mu_\alpha(J, A) = 0 \Longleftrightarrow \text{solution to Kähler–Yang–Mills equations}$$

For $\alpha > 0$, $(\mathscr{P}, \omega_\alpha)$ has a canonical $\widetilde{\mathscr{G}}$-invariant Kähler structure

Moduli $\mathcal{M}_\alpha := \{\text{solutions to Kähler–Yang–Mills equations}\}/\widetilde{\mathscr{G}}$ is Kähler for $\alpha > 0$

Remarks of KYM equations

- We recover the Hermitian–Yang–Mills equations, while the equation $S_{\omega,J} =$ constant (Yau–Tian–Donaldson theory) is deformed
- The coupling term in the second equation comes precisely from the non-triviality of the extension defining $\widetilde{\mathscr{G}}$.
- Equations 'decouple' for $\dim_{\mathbb{C}} M = 1$ (as $F_h^2 = 0$ in this case)
- Solutions to the Kähler–Yang–Mills equations are absolute minima of a certain **Calabi–Yang–Mills** functional

Remarks on KYM equations

Programme: Study existence of solutions

- **Very hard problem**: In general, this is a system of coupled fourth-order fully non linear partial differential equations!
- **Motivation**: Analytic approach to the algebraic geometric problem of studying the moduli space classifying pairs (X, E) consisting of a projective variety and a holomorphic vector bundle.
- In the paper [AGG2013] we give some existence results for small α, by perturbation from constant scalar curvature Kähler metrics and Hermitian Yang–Mills connections.
- More concrete and interesting solutions — over a polarised threefold not admitting any constant scalar curvature Kähler metric — were obtained by **Keller and Tønnesen–Friedman** (2012).
- **Garcia-Fernandez–Tipler** (2013) added new examples to this short list by simultaneous deformation of the complex structures of M and E.

Remarks on KYM equations

- In [AGG2013], we also study **obstructions** for the existence of solutions, generalizing the **Futaki invariant**, the **Mabuchi** K**-energy** and **geodesic stability** (Chen, Donaldson) that appear in the constant scalar curvature theory
- **But** a general existence theorem is still missing...
- Consider a **simpler situation** where there is a **group of symmetries** acting on the picture: **dimensional reduction**
- One of the simplest cases to consider is $M = X \times \mathbb{P}^1$, where X is a Riemann surface, with SU(2) as group of symmetries
- **This study was initiated in**:

  L. Álvarez-Cónsul, M. García-Fernández and O. García-Prada, Gravitating vortices, cosmic strings and the Kähler–Yang–Mills equations, *Comm. Math. Phys.* **351** (2017) 361–385.

  Building upon:

  O. García-Prada, Invariant connections and vortices, *Comm. Math. Phys.*, **156** (1993) 527–546.

Abelian vortices on a compact Riemann surface

- X compact Riemann surface
- ω_X Kähler form on X (conveniently normalised)
- $L \to X$ holomorphic line bundle over X
- $\varphi \in H^0(X, L)$ holomorphic section of L

Vortex equations

for a Hermitian metric h on L:

$$i\Lambda F_h + |\varphi|_h^2 - \tau = 0$$

- $F_h \in \Omega^2(X)$ curvature of Chern connection of h on L
- $\Lambda F_h \in C^\infty(X)$ contraction of F_h with ω_X
- $|\cdot|_h \in C^\infty(X)$ positive norm on L associated to h
- $\tau \in \mathbb{R}$ real parameter

Abelian vortices on a compact Riemann surface

- **Physics**: Ginzburg–Landau theory of superconductivity
  Here F_h is the magnetic field and $|\varphi|_h^2$ is the density of Cooper pairs
- The Lagrangian depends on a parameter λ
- For the 'Bogomol'nyi phase' $\lambda = 1$, The Euler–Lagrange equations are equivalent to the vortex equations

Integrating (i.e. applying $\int_X (-)\omega_X$ to the equation) we obtain

$$2\pi \deg L + \|\varphi\|_{L^2}^2 = \tau \operatorname{vol}(X)$$

Normalising $\operatorname{vol}(X) = 2\pi$, this implies

$$\deg L \leq \tau$$

Abelian vortices on a compact Riemann surface

Theorem

Existence of solutions to vortex equations $\iff$ $\deg L \leq \tau$

- If $\varphi = 0$, by Hodge theory: Existence $\iff$ $\deg L = \tau$
- For $\varphi \neq 0$ several proofs:
- **Noguchi** (1987, $\tau = 1$): direct proof using **continuity method**.
- **Bradlow** (1990): reduces to **Kazdan–Warner equation** in Riemannian geometry.
- **GP** (1991, PhD Thesis): **dimensional reduction of Hermitian Yang–Mills equations**

Hermitian Yang–Mills equations

- (M, ω_M) compact Kähler manifold with $\dim_{\mathbb{C}} M = n$ (normalisation $\mathrm{vol}(M) = 2\pi$)
- $E \to M$ holomorphic vector bundle over M

Hermitian Yang–Mills equations

for a **Hermitian metric** h on E:

$$i\Lambda F_h = \mu \, \mathrm{Id}_E$$

- Taking traces in the equation and $\int_M (-) \, \mathrm{dvol}_M$:

$$\mu = \mu(E) = \frac{\deg_{\omega_M}(E)}{\operatorname{rank} E}$$

where

$$\deg_{\omega_M}(E) = \int_M c_1(E) \omega_M^{n-1}.$$

Hermitian Yang–Mills equations

Definition (Mumford–Takemoto)

- E is **stable** if $\mu(E') < \mu(E)$ for every coherent subsheaf $0 \neq E' \subsetneq E$.
- E is **polystable** if $E \cong \oplus E_i$ with E_i stable of the same slope.

Theorem (Donaldson–Uhlenbeck–Yau (1986–87)

Existence of solutions to Hermitian Yang–Mills equations on E $\iff$ E polystable

Irreducible solution $\iff$ E stable

- **Come back to a pair** (L, φ) over compact Riemann surface X.
- Associate a **rank** 2 **holomorphic vector bundle** E over $X \times \mathbb{P}^1$:

$$0 \to p^*L \to E \to q^*\mathcal{O}_{\mathbb{P}^1}(2) \to 0$$

$p : X \times \mathbb{P}^1 \to X$ and $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$ natural projections

- Extensions as above are parametrized by

$$H^1(X \times \mathbb{P}^1, p^*L \otimes q^*\mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(X, L) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(X, L)$$

by Künneth formula, and Serre duality $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^* \cong \mathbb{C}$

SU(2)-**action**:

- SU(2) acts on $X \times \mathbb{P}^1$: Trivially on X and $\mathbb{P}^1 = \mathrm{SU}(2)/\,\mathrm{U}(1)$
- SU(2)-action can be lifted to E: trivially on p^*L and standard on $q^*\mathcal{O}_{\mathbb{P}^1}(2)$
- Trivial action of SU(2) on $H^0(X, L) \implies$
  E is a SU(2)-**equivariant** holomorphic vector bundle
- SU(2)-**invariant Kähler metric** on $X \times \mathbb{P}^1$

$$\omega_\tau = p^*\omega_X \oplus \frac{4}{\tau} q^*\omega_{\mathbb{P}^1}^{FS}$$

$\tau > 0$; $\omega_{\mathbb{P}^1}^{FS}$ Fubini–Study metric (volume normalised to 2π)

Relation of Hermitian–Yang–Mills equations to vortices

Theorem (existence of vortices on $L \longrightarrow X$)

$$\begin{array}{ccc} \text{solution to } \tau\text{-vortex on } X & \Longleftrightarrow & \tau > \deg L \\ \Big\Updownarrow \text{GP} & & \Big\Updownarrow \text{GP} \\ \text{SU(2)-inv. } \omega_\tau\text{-HYM metric} & \overset{\text{Donaldson–Uhlenbeck–Yau}}{\Longleftrightarrow} & E \ \omega_\tau\text{-polystable} \end{array}$$

- vortices on $X \quad \longleftrightarrow \quad$ SU(2)-**dimensional reduction** of ASD instantons on $X \times \mathbb{P}^1$.
- Generalises results of **Witten** (1977) for vortices on the hyperbolic real plane and **Taubes** for vortices on $\mathbb{R}^2$ (1980).

Moduli space of vortices

- Fix **smooth line bundle** L of degree $\deg L = N$ over X and Hermitian metric h on L
- **Vortex equations**: solving for a unitary connection A on (L, h) and a smooth section φ of L satisfying

$$\begin{aligned} i\Lambda F_A + |\varphi|_h^2 - \tau &= 0 \\ \bar{\partial}_A \varphi &= 0 \end{aligned}$$

- Solutions are called **vortices**.
- A gauge equivalence class of solutions is determined by the zeros of φ: **effective divisor**.

Moduli space of vortices

Let $\mathscr{V}_\tau$ be the **moduli space of vortices on** L for $\tau > N$. Then

$$\mathscr{V}_\tau \cong \mathrm{Sym}^N(X)$$

Dimensional reduction of KYM equations: Gravitating vortex equations

- **Proposition**: Let E be the SU(2)-equivariant rank 2 holomorphic vector bundle over $X \times \mathbb{P}^1$ defined by (L, φ). A SU(2)-**invariant solution** to the Kähler–Yang–Mills equations on $E \to X \times \mathbb{P}^1$ is equivalent to a solution of the following equations:

Gravitating vortex equations

for a metric g on X and a Hermitian metric h on L

$$i\Lambda_g F_h + |\varphi|_h^2 - \tau = 0$$
$$S_g + \alpha(\Delta_g + \tau)(|\varphi|_h^2 - \tau) = c$$

Gravitating vortex = solution of the gravitating vortex equations

- $\tau > 0$, $\alpha > 0$ real parameters
- $c \in \mathbb{R}$ is determined by the topology

Dimensional reduction: Gravitating vortex equations

- The first equation

$$i\Lambda_g F_h + |\varphi|_h^2 - \tau = 0$$

is the **abelian vortex equation**

- Integrating it, we obtain

$$2\pi \deg L + \|\varphi\|_{L^2}^2 = \tau \,\mathrm{vol}_g(X)$$

Assuming $\varphi \neq 0$, this implies that in order to have solutions we must have

$$\deg L \leq \frac{\tau \,\mathrm{vol}_g(X)}{2\pi}$$

Theorem (Noguchi, 1987; Bradlow, 1990; GP, 1991)

Existence of solutions to the vortex equation
$\iff \quad \deg L \leq \frac{\tau \,\mathrm{vol}_g(X)}{2\pi}$

- Coming back to the gravitating vortex equations, by integrating the first equation we have

$$\int_X (\varphi|_h^2 - \tau) = \frac{2\pi \deg(L)}{\mathrm{vol}_g(X)}$$

- On the other hand

$$\int_X \Delta_g(|\varphi|_h^2 - \tau) = 0 \quad \text{and} \quad \int_X S_g = \frac{2\pi\chi(X)}{\mathrm{vol}_g(X)}$$

- With this, and integrating the second equation we have

$$c = \frac{2\pi}{\mathrm{vol}_g(X)}(\chi(X) - \alpha\tau \deg(L))$$

In particular we have

$$c \geq 0 \Longrightarrow X = \mathbb{P}^1$$

Physics: cosmic strings and topological defects

- When $c = 0$ our equations are known in the physics literature as **Einstein–Bogomol'nyi equations** and their solutions are called **Nielsen–Olesen cosmic strings**
- $\alpha = 2\pi G$, $G > 0$ is universal gravitation constant

Mathematical Physics literature: Linet (1988), Comtet–Gibbons (1988), Spruck–Yisong Yang (1995), Yisong Yang (1995) ...

More on gravitating vortices in **Mario Garcia-Fernandez's** talk!