Instanton bundles of high rank on Fano threefolds

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Instanton bundles on $\mathbb{P}^3$

We consider the projective space $\mathbb{P}^3$.

Definition (i)

An instanton bundle of charge k on $\mathbb{P}^3$ is a stable rank 2 vector bundle E with Chern classes $c_1(E) = 0, \ c_2(E) = k$ and satisfying:

$$H^1(E(-2)) = 0.$$

Aim: extend the notion of instanton to vector bundles of arbitrary rank on a Fano 3-fold X with $Pic(X) = H_X \cdot \mathbb{Z}$.

Note: $-K_X \sim i_X H_X = (2q_X + r_X)H_X$ (i_X is the **index** of X).

Instantons on Fano threefolds of Picard rank one

Definition

An instanton bundle is a vector bundle E on X such that:

- *E is Gieseker semi-stable;*
- $h^1(E(n)) = 0,\ \forall n \leq -q_X,\ \ h^2(E(n)) = 0,\ \forall n \geq -q_X;$
- $ch(E) = ch(E^*(-r_X))$.

$c_2(E) = k$ is referred to as the **charge** of E.

Question: For which values of (r, k) we have the existence **stable** instantons of rank r and charge k?

Main steps:

- Define the minimal charge $k_{0,r}$;
- Prove that we indeed have existence $\forall k \geq k_{0,r}$.

Rank 2 instanton bundles

We have 2 main ways to prove existence of rank 2 instantons.

- **Constructive:** Direct construction of monads (for $H^3(X) = 0$); Serre's correspondence: (C, e) with $C \subset X$ l.c.i. curve and $e \in H^0(\omega_C(r_X + i_X - 2m))$ nowhere vanishing $\Leftrightarrow$:

$$0 \to \mathcal{O} \to E(m) \to \mathcal{I}_C(2m - r_X) \to 0$$

- **Via deformation:** Start from (E, l, ϕ):
  - E a stable rank 2 instanton bundle with $Ext^2(E, E) = 0$,
  - $l \subset X$ a line such that $\mathcal{N}_{l/X} \simeq \mathcal{O}_l \oplus \mathcal{O}_l(-q_X) \simeq E|_l$,
  - $\phi : E \to \mathcal{O}_l(-q_X)$ an epimorphism.

  We show that $F := \ker(\phi)$:

$$0 \to F \hookrightarrow E \to \mathcal{O}_l(-q_X) \to 0,$$

  deforms to a locally free sheaf.

Existence of rank 2 instanton bundles: known results

Combining these 2 methods we get the following:

Theorem

Let X be a Fano threefold of index $i_X \in \{2, 3, 4\}$. Then $\forall\ k \geq 1$ if $i_X = 3, 4$ and $\forall\ k \geq 2$ for $i_X = 2$, there exists a stable rank 2 instanton bundle of charge k on X.

For $i_X = 1$ we only know existence in the following cases:

Theorem

Let X be a Fano threefold of index $i_X = 1$ and genus $g(X) \geq 3$.

- *if X contains an ordinary line $\Rightarrow \exists$ rank 2 stable instanton bundles of charge k on X $\forall\ k \geq \lceil \frac{g(X)}{2} \rceil + 1$;*
- *if X does not contains ordinary lines $\Rightarrow \exists$ rank 2 stable instaton bundles of charge $k \in \{\lceil \frac{g(X)}{2} \rceil + 1, \ldots g(X) + 3\}$.*

Existence of high rank instantons on $\mathbb{P}^3$

Lemma

E_k^r Gieseker semistable instanton bundle on $\mathbb{P}^3$ of $c_2(E) = k$ and rank $r = 2m$, (resp. rank $r = 2m+1$), $m \geq 1 \Rightarrow k \geq m$ (resp. $k \geq m+1$).

This is simply due to $\chi(E_k^r) = r - 2k$ (and global sections always prevent stability!!)

Theorem

$\forall\, r \geq 2$, $\forall\, k \geq \lfloor \frac{r}{2} \rfloor$ if r even and $\forall\, k \geq \lfloor \frac{r}{2} \rfloor + 1$ if r odd, $\exists$ a μ-stabe and unobstructed (i.e. $\mathrm{Ext}^2(E,E) = 0$) instanton bundle E on $\mathbb{P}^3$ of rank r, $c_2(E) = k$ and such that $h^0(E(1)) > 0$.
For $s \in H^0(E(1))$ general, $\operatorname{coker} \mathcal{O}(-1) \xrightarrow{s} E$ is a unobstructed μ-stable sheaf.

Existence of high rank instantons on $\mathbb{P}^3$: idea of the proof

Idea of the proof:
We work by induction.

- the **t'Hooft bundles** shows that the theorem holds for $r = 2$:

$$0 \to \mathcal{O}(-1) \to E_k^2 \to \mathcal{I}_C(1) \to 0$$

  for C the union of $k+1$ disjoint lines.

- for $r > 2$, any instanton bundle E_k^r of charge $k > k_{0,r}$ always satisfies $ext^1(E_k^r, \mathcal{O}) > 0$ hence we get the existence of a strictly μ-stable instanton bundle E_k^{r+1} fitting in:

$$0 \to \mathcal{O} \to E_k^{r+1} \to E_k^r \to 0.$$

Existence of high rank instantons on $\mathbb{P}^3$: idea of the proof.

- The bundle $E_k^{r+1}(1)$ always admits a global section giving rise to a commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & \mathcal{O}(-1) & = & \mathcal{O}(-1) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & E_k^{r+1} & \longrightarrow & E_k^r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & F_k^{r+1} & \longrightarrow & F_k^r & \longrightarrow & 0
\end{array}
$$

- we show that a general deformation of F_k^{r+1} has no global sections $\Rightarrow E_k^{r+1}$ deforms to a μ-stable instanton bundle.

Existence of high rank instantons: $i_X = 2$

Lemma

Let E_k^r be a μ-stable rank r instanton bundle of charge k on a Fano 3-fold of index 2. Then $k \geq r$.

The method used for $\mathbb{P}^3$ adapt for the case of index 2.
We first need to construct a family of stable rank 2 instanton bundles E on X such that $h^0(E(1)) > 0$.
These are obtained from elliptic curves on X (via Serre).

Proposition

For all $k \geq 2$, $\exists$ a generically smooth $2(d_X + k)$ dimensional component $\mathcal{H}_{d_X+k}$ of $\mathrm{Hilb}_{(d_X+k)t}(X)$, whose generic point is a smooth, non-degenerate and unobstructed curve.

Existence of high rank instantons: $i_X = 2$

Proposition

For all $k \geq 2$ $\exists$ an unobstructed μ-stable rank 2 instanton bundle E_2^k fitting in a short exact sequence

$$0 \to \mathcal{O}(-1) \to E \to \mathcal{I}_C(1) \to 0 \quad (1)$$

with C a smooth, non-degenerate and unobstructed curve in $\mathcal{H}_{d_X+k}$.

Theorem

$\forall$ $r \geq 2$ and $\forall$ $k \geq 0$ there exists an unobstructed μ-stable instanton bundle E on X of rank r, charge $c_2(E) = r + k$ and such that $h^0(E(1))$. $\forall$ $s \in H^0(E(1))$, $\operatorname{coker}(s)$ is a unobstructed μ-stable torsion free sheaf.

Instantons of high rank on quadrics

Let us consider a 3-dimensional quadric hypersurface $Q \subset \mathbb{P}^4$.

Theorem

A rank r instanton of charge k is the cohomology of a monad of the form:

$$0 \to I_- \otimes \mathcal{O}(-1) \to W \otimes \mathcal{S} \to I_+ \otimes \mathcal{O}(1) \to 0$$

with $I_- \simeq I_+ \simeq \mathbb{C}^{k-1}$, $W \simeq \mathbb{C}^{\frac{r}{2}+k-1}$ and $\mathcal{S}$ the spinor bundle on Q.

Lemma

Let E_k^r be a μ-stable instanton bundle of rank $r = 2m$, $m \geq 2$. on Q. Then $k \geq m + 1$

This is shown by looking at the monads: if $c_2(E_k^r) = m$, $E_k^r \simeq \mathcal{S}^m$.

Instantons of high rank on quadrics

Theorem

For all $r = 2m$, $m \geq 1$ and $\forall k \geq m+1$ if $m > 1$, $\forall\, k \geq 1$ for $m = 1$, $\exists$ a μ-stable unobstructed rank r instanton of charge k.

Idea of the proof:

- For $m = 1$ construct **t'Hooft** bundles E_k^r of charge k:
$$0 \to \mathcal{O}(-1) \to E_k^2 \to \mathcal{I}_C \to 0$$
- If E_k^r is a unobstructed μ-stable instanton bundle of rank $2m$ and charge $k > m+1$, $m > 1$, we then have $ext^1(E_k^r, \mathcal{S}) > 0$ hence we get the existence of a strictly μ-semistable unobstructed instanton bundle:
$$0 \to \mathcal{S} \to E_{k+1}^{r+2} \to E_k^r \to 0$$
- We show that E_{k+1}^{r+2} deform to a μ-stable instanton bundle.

Instantons of high rank: $i_X = 1$

Let us finally consider a Fano 3-fold X of index 1 and genus $g(X) \geq 3$ (so that H_X is very ample.) We first extend the existence results for the rank 2 cases. Set $k_{0,2} := \lceil \frac{g(X)}{2} \rceil + 1$.

Proposition

Let X be a Fano 3-fold X of index 1 and genus $g(X) \geq 3 \Rightarrow \forall\, k \geq k_{0,2}$ there exists a μ-stable and unobstructed rank 2 instanton bundle of charge k.

Idea of the proof We prove that given a elementary transformation:

$$0 \to F \to E \to \mathcal{O}_C(-p) \to 0$$

with E a rank 2 μ-stable unobstructed instanton bundle, C a conic non jumping for E, the sheaf F always deforms to a vector bundle.

Instantons of high rank: $i_X = 1$

The "expected" minimal charge $k_{0,2m}$ is $mk_{0,2}$ if $g(X)$ even and $mk_{0,2} + 1$ if $g(X)$ odd. We can only prove the existence for $k \geq k_{0,2m} + 1$

Theorem

Let X be a Fano threefold of index $i_X = 1$ and $g(X) \geq 3 \Rightarrow$:

- *$\forall k \geq k_{0,2}$, $\exists$ rank 2 μ-stable unobstructed instanton bundles of charge k,*
- *$\forall$ $k \geq m \cdot k_{0,2} + 1$ if $g(X)$ even and $k \geq m \cdot k_{0,2} + 2$ if $g(X)$ odd, $\exists$ a μ-stable unobstructed instanton bundle of rank $2m$ and charge k.*

Thanks for the attention