Flows of $\mathrm{SU}(2)$-structures and beyond

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Motivation

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Want to address this problem for an H-structure with $H \subset O(n)$.

Berger's classification

My main interest is when H belongs to Berger's list:

Dim	Holonomy group	Parallel tensor	Geometry
$2n$	$\mathrm{U}(n)$	ω, J	Kähler
$2n$	$\mathrm{SU}(n)$	ω, Ψ	Calabi-Yau
$4n$	$\mathrm{Sp}(n)$	$\omega_1, \omega_2, \omega_3$	HyperKähler
$4n$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	Ω	Quaternion-Kähler
7	G_2	φ	G_2
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Holonomy reduction is equivalent to $\nabla\eta = 0$, where $\mathrm{Stab}(\eta) \cong H \subset SO(n)$.
Interested in finding solutions by means of geometric flow i.e.

$$\frac{\partial}{\partial t}\eta = \cdots$$

One can also consider Sasaki, 3-Sasaki, $\mathrm{PSU}(3)$,... structures

General framework for flows of H-structures

Given a 2-tensor $A \in \mathfrak{gl}(\mathfrak{n}, \mathbb{R}) \cong T^* \otimes T^*$ we can act by endomorphism $\diamond$ on any tensor α i.e. $(A \diamond \alpha)_{ij\dots k} = A_{ia}\alpha_{aj\dots k} + \cdots + A_{ka}\alpha_{ij\dots a}$.

[Fadel-Loubeau-Moreno-Sá Earp 22], [Karigiannis 08]

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Note that if $A \in \mathfrak{h}$ then $A \diamond \eta = 0$, so we need

$$A \in \mathrm{End}(\mathbb{R}^n)_{\mathfrak{h}} := \mathrm{End}(\mathbb{R}^n)/\mathfrak{h} = S^2 \oplus \mathfrak{h}^{\perp}.$$

Invariant quantities: rep theory

We now know which spaces A has to belong to, but how do we choose it?

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Bryant 03

Given $H \subset O(n)$, the space of kth order invariants $V_k(\mathfrak{h})$ is implicitly given by

$$V_k(\mathfrak{h}) \oplus (\Lambda^1 \otimes S^{k+1}) = (S^2 \oplus \mathfrak{h}^\perp) \otimes S^k,$$

where $S^k = S^k(\Lambda^1)$ and $\mathfrak{h}^\perp = \Lambda^2/\mathfrak{h}$.

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Proof.

$V_1(\mathfrak{h}) = \Lambda^1 \otimes \mathfrak{h}^\perp$ is the space of intrinsic torsion. On the other hand $V_2(\mathfrak{h})$ is spanned by ∇T and Rm but these overlap in $\mathfrak{h}^\perp \otimes \Lambda^2$ due to a Bianchi type identity: $\nabla T \diamond \eta = \nabla^2 \eta + \mathrm{l.o.t}$. Can iterate this argument for other $V_k(\mathfrak{h})$ using differential Bianchi identity. □

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Let us illustrate this explicitly in a few cases!

Simplest case i.e. $H = \mathrm{O}(n)$

Cartan, Weyl

$$V_1(\mathfrak{so}(n)) = 0$$
$$V_2(\mathfrak{so}(n)) = 1\mathbb{R} \oplus 1S^2_0(\mathbb{R}^n) \oplus W$$
$$\mathrm{End}(\mathbb{R}^n)_{\mathfrak{so}(n)} = 1\mathbb{R} \oplus 1S^2_0(\mathbb{R}^7)$$

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$$1 \times 1 + 1 \times 1 = 2$$

There is a 2-parameter of $\mathrm{O}(n)$ flows!

$$\partial_t(g) = \lambda_1 \mathrm{Scal}(g)g + \lambda_2 \mathrm{Ric}(g).$$

The G_2 case

Bryant 03, Dwivedi-Karigiannis-Gianniotis 23

$$V_1(\mathfrak{g}_2) = \mathbb{R} \oplus \mathbb{R}^7 \oplus S^2_0(\mathbb{R}^7) \oplus \mathfrak{g}_2$$

$$V_2(\mathfrak{g}_2) = 1\mathbb{R} \oplus 2\mathbb{R}^7 \oplus 3S^2_0(\mathbb{R}^7) \oplus V_{0,1} \oplus 2V_{1,1} \oplus V_{0,2} \oplus V_{3,0}$$

$$\mathrm{End}(\mathbb{R}^7)_{\mathfrak{g}_2} = 1\mathbb{R} \oplus 1\mathbb{R}^7 \oplus 1S^2_0(\mathbb{R}^7)$$

$$1 \times 1 + 2 \times 1 + 3 \times 1 = 6$$

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There is a 6-parameter of G_2 flows to second order!

$$\partial_t(\varphi) = (\lambda_1 \mathrm{Scal}(g) g + \lambda_2 \mathrm{Ric}(g) + \lambda_3 W_{27} + \lambda_4 \mathcal{L}_{\tau_1} g + \lambda_5 \mathrm{div}(T) + \lambda_6 \mathrm{div}(T^t)) \diamond \varphi.$$

Recall that $T \in \Lambda^1 \otimes \mathfrak{g}_2^\perp \subset \Lambda^1 \otimes \Lambda^2$ is defined by $T \diamond \varphi = \nabla \varphi$.

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Rmk: For $\partial_t(\varphi) = \Delta_\varphi \varphi$ AND $d\varphi_t = 0$, $\lambda_2 = -1$ and $\lambda_i = 0$ otherwise.

The Spin(7) case

F20, Dwivedi 24

$$V_1(\mathfrak{spin}(7)) = \mathbb{R}^8 \oplus \mathbb{R}^{48}$$
$$V_2(\mathfrak{spin}(7)) = 1\mathbb{R} \oplus 1\mathbb{R}^7 \oplus 2S^2_0(\mathbb{R}^8) \oplus V_{0,1,0} \oplus V_{1,1,0} \oplus V_{2,0,0} \oplus V_{0,2,0} \oplus V_{1,0,2}$$
$$\mathrm{End}(\mathbb{R}^8)_{\mathfrak{spin}(7)} = 1\mathbb{R} \oplus 1\mathbb{R}^7 \oplus 1S^2_0(\mathbb{R}^8)$$

$$1 \times 1 + 2 \times 1 + 1 \times 1 = 4$$

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There is a 4-parameter of $\mathrm{Spin}(7)$ flows to second order!

$$\partial_t(\Phi) = (\lambda_1 \mathrm{Scal}(g)g + \lambda_2 \mathrm{Ric}(g) + \lambda_3 \mathcal{L}_{T_1} g + \lambda_4 \mathrm{div}(T)) \diamond \Phi.$$

The $\mathrm{SU}(3)$ case

Bedulli-Vezzoni 07

$$V_1(\mathfrak{su}(3)) = 2\mathbb{R} \oplus 2\mathbb{R}^6 \oplus 2\mathfrak{su}3 \oplus \mathbb{R}^{12}$$

$$V_2(\mathfrak{su}(3)) = 3\mathbb{R} \oplus 4\mathbb{R}^6 \oplus 5\mathfrak{su}3 \oplus 4\mathbb{R}^{12} \oplus 3V_{2,1} \oplus V_{2,2} \oplus V_{3,0}$$

$$\mathrm{End}(\mathbb{R}^6)_{\mathfrak{su}(3)} = 2\mathbb{R} \oplus 1\mathbb{R}^6 \oplus 1\mathfrak{su}3 \oplus 1\mathbb{R}^{12}$$

$$3 \times 2 + 4 \times 1 + 5 \times 1 + 4 \times 1 = 19$$

There is a 19-parameter of $\mathrm{SU}(3)$ flows to second order!

In fact out of the $3 + 4 + 5 + 4 = 16$ invariants, 7 are computable only from curvature: 3 of which come from Ricci and 4 from Weyl!

The $\mathrm{SU}(2)$ case

F.-Sá Earp 24

$$V_1(\mathfrak{su}(2)) = 3\mathbb{R}^4$$

$$V_2(\mathfrak{su}(2)) = 9\mathbb{R} \oplus 12\mathbb{R}^3 \oplus \mathbb{R}^5$$

$$\mathrm{End}(\mathbb{R}^4)_{\mathfrak{su}(2)} = 4\mathbb{R} \oplus 3\mathbb{R}^3$$

$$9 \times 4 + 12 \times 3 = 72$$

There is a 72-parameter of $\mathrm{SU}(2)$ flows to second order!
Let's be more explicit in this case.

The $\mathrm{SU}(2)$ case

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Rmk: $V_2(\mathfrak{sp}(2)) = 13\mathbb{R} \oplus 27\mathfrak{sp}(2) \oplus 24\mathbb{R}^5 \oplus \cdots$ so it gets harder in higher dimension....

First order invariants

The intrinsic torsion $T \in \Lambda^1 \otimes \mathfrak{su}(2)^\perp \subset \Lambda^1 \otimes \Lambda^2$ is defined by

$$T(\cdot) \diamond \boldsymbol{\omega} = \nabla.\boldsymbol{\omega},$$

where $\boldsymbol{\omega} := (\omega_1, \omega_2, \omega_3)$, but $\mathfrak{su}(2)^\perp \cong \Lambda^2_+$, so we have

$$T = a_1 \otimes \omega_1 + a_2 \otimes \omega_2 + a_3 \otimes \omega_3 \in 3\mathbb{R}^4 \cong V_1(\mathfrak{su}(2))$$

Explicitly,

$$a_1 = \frac{1}{2}(- * d\omega_1 - J_2 * d\omega_2 + J_3 * d\omega_3)$$
$$a_2 = \frac{1}{2}(- * d\omega_2 - J_3 * d\omega_3 + J_1 * d\omega_1)$$
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Next we want: Second order invariants $V_2(\mathfrak{su}(2))$!

Second order invariants

Recall that

$$V_2(\mathfrak{su}(2)) = 9\mathbb{R} \oplus 12\mathbb{R}^3 \oplus \mathbb{R}^5.$$

F-Sá Earp

All second-order invariants of an $\mathrm{SU}(2)$-structure can be expressed in terms of the following linearly independent terms:

1. The 9 functions $\delta a_1, \delta a_2, \delta a_3$ and $g(da_i, \omega_j)$, where $1 \leq i \leq j \leq 3$.
2. The 12 anti-selfdual forms $\pi^2_-(da_i)$, where $i = 1, 2, 3$, and $S^2_{0,j}(\nabla a_i)$, where $i, j = 1, 2, 3$.
3. The anti-selfdual Weyl curvature $W^- \in S^2_0(\Lambda^2_-)$.

In particular, this implies that Ric and W^+ are expressible in terms of (1) and (2) + lower order terms in $S^2(V_1(\mathfrak{su}(2)))$.

Explicit formulae: Self-dual Weyl

$$
\begin{aligned}
W^+ =& -\frac{1}{24}\Big(2g(da_1,\omega_1) - g(da_2,\omega_2) - g(da_3,\omega_3) \\
&+ g(J_3a_1,a_2) - g(J_2a_1,a_3) - 2g(J_1a_2,a_3)\Big)(2\omega_1^2-\omega_2^2-\omega_3^2) \\
&- \frac{1}{8}\Big(g(da_2,\omega_2) - g(da_3,\omega_3) + g(J_3a_1,a_2) + g(J_2a_1,a_3)\Big)(\omega_2^2-\omega_3^2) \\
&- \frac{1}{4}(g(da_2,\omega_1) + g(J_1a_1,a_3))(\omega_1\odot\omega_2) \\
&- \frac{1}{4}(g(da_3,\omega_2) + g(J_2a_2,a_1))(\omega_2\odot\omega_3) \\
&- \frac{1}{4}(g(da_1,\omega_3) + g(J_3a_3,a_2))(\omega_3\odot\omega_1) \in S^2_0(\Lambda^2_+).
\end{aligned}
$$

Explicit formulae: Ricci

$$\mathrm{Scal}(g) = -2\Big(\sum_{i=1}^{3} g(da_i, \omega_i) + g(J_1(a_1), J_2(a_2)) + g(J_1(a_1), J_3(a_3)) + g(J_2(a_2), J_3(a_3))\Big).$$

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$$\mathrm{Ric}_0(g) = P_1(\pi^2_-(\Phi_1)) + P_2(\pi^2_-(\Phi_2)) + P_3(\pi^2_-(\Phi_3)),$$

where recall $P_i : \Lambda^2_- \xrightarrow{\simeq} S^2_{0,i}$ and

$$\Phi_1 := da_1 + \frac{1}{4}(J_2(a_2) \wedge J_2(a_3) - J_1(a_1) \wedge J_2(a_3) + J_2(a_2) \wedge a_1) - \frac{1}{4}(J_1(a_1) \wedge J_3(a_2) + J_3(a_3) \wedge a_1) - \frac{3}{4}(J_3(a_3) \wedge J_3(a_2)),$$

and similar expressions hold for Φ_2, Φ_3.

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and similar expressions hold for Φ_2, Φ_3.

Let's go back to $\mathrm{SU}(2)$-flows

SU(2) flows

Any $\mathrm{SU}(2)$-flow can be expressed as

$$\partial_t(\boldsymbol{\omega}) = A \diamond \boldsymbol{\omega}$$

where $A = f_0 g + B_{0,1} + B_{0,2} + B_{0,3} + f_1\omega_1 + f_2\omega_2 + f_3\omega_3$

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$$\begin{aligned}
\partial_t\omega_1 &= +f_0\omega_1 + f_3\omega_2 - f_2\omega_3 + \tilde{B}_{0,1} \\
\partial_t\omega_2 &= -f_3\omega_1 + f_0\omega_2 + f_1\omega_3 + \tilde{B}_{0,2} \\
\partial_t\omega_3 &= +f_2\omega_1 - f_1\omega_2 + f_0\omega_3 + \tilde{B}_{0,3}
\end{aligned}$$

Recall that there is a 72-parameter of $\mathrm{SU}(2)$-flows.

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$$\begin{aligned}
\partial_t\omega_1 &= +f_0\omega_1 + f_3\omega_2 - f_2\omega_3 + \tilde{B}_{0,1}\\
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\end{aligned}$$

Recall that there is a 72-parameter of $\mathrm{SU}(2)$-flows.
But there are some flows which are more natural than other namely those that arise gradient of some functionals.

Quadratic functionals

$$S^2(V_1(\mathfrak{su}(2))) = 15\mathbb{R} \oplus 21\mathbb{R}^3$$

So there are 15 quadratic functionals in T. These are explicitly given by

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By comparison:

$$S^2(V_1(\mathfrak{g}_2)) = 4\mathbb{R} \oplus \cdots$$
$$S^2(V_1(\mathfrak{spin}(7))) = 2\mathbb{R} \oplus \cdots$$
$$S^2(V_1(\mathfrak{su}(3))) = 11\mathbb{R} \oplus \cdots$$

Rmk: One can also consider functionals coming the $\mathbb{R}$ components of $V_2(\mathfrak{h})$ e.g. $\int_M \mathrm{Scal}(g)\,\mathrm{vol}$

SU(2) flows

Consider the 2-parameter family given by

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SU(2) flows

F- Sá Earp

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has short time existence and uniqueness for $9 - 4\sqrt{5} < \lambda_2 + \frac{\lambda_1}{2} < 9 + 4\sqrt{5}$.

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Upshot: Symmetric part is Ricci DeTurck flow (which is strongly parabolic!), so suffices to linearise skewsymmetric part and compute the condition when principal symbol of $L_{\mathcal{P}}$ is invertible: this gives the range.

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S3: Pulling back by the flow generated by X yields short time existence for original flow: $\partial_t(\boldsymbol{\omega}) = \mathcal{Q}(\boldsymbol{\omega})$.
S4: Uniqueness: given 2 solutions, we can pullback by the diffeomorphisms generated by λV_i to get 2 Ricci like flows. Use harmonic map flow to get another pair of diffeomorphisms and show that pullback flow is parabolic (hence unique). Then since V_i depend on $\boldsymbol{\omega}_i$, one can argue that they induce the same diffeomorphism.

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2. The Lie derivative term can more generally be set to $\sum_{i,j} \lambda_{ij} \mathcal{L}_{J_i a_j} g$
3. Questions: What l.o.t to choose? Are there flows preserving torsion classes (or combinations of those) e.g. Laplacian flow? Long time existence, singularities, ...

Thank you for listening!