Flats in the space of Kähler metrics

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Metric structure on $\mathcal{H}_L$.

By work of **Mabuchi** '82, there is a natural metric on $\mathcal{H}_L$, where

$$d_2(\phi_0, \phi_1)^2 := \inf_{\{\phi_t\}} \int_0^1 \int_X |\dot{\phi}_t|^2 (\omega + dd^c\phi_t)^n,$$

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(Semmes, Donaldson '90s: particular geometric importance in the cscK problem.)

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$\mathcal{H}_L \subset \mathcal{H}_L^\infty$ is a dense embedding.

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In a space of negative curvature, it is natural to study its **boundary at infinity**: equivalence classes of geodesic rays $[0, \infty) \ni t \mapsto \phi_t, \psi_t$ with

$$\frac{d_2(\phi_t, \psi_t)}{t} \to_{t\to\infty} 0.$$

Quantisation.

A metric $\phi \in \mathcal{H}_L^\infty$ induces a Hermitian norm on the space of sections $\Gamma(X, L^{\otimes k})$ given by

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1. controlled growth as $k \to \infty$;
2. "almost" submultiplicativity: if s is a section of $L^{\otimes k}$, t of $L^{\otimes m}$, then

$$\|s \cdot t\|_{\phi^{\otimes k+m}} \text{"} \leq \text{"} \|s\|_{\phi^{\otimes k}} \|t\|_{\phi^{\otimes m}}.$$

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By general quantisation results (Bouche, Tian, Catlin, Zelditch, ...), one can recover ϕ from the data of this sequence of norms.

We can thus first look at the boundary at infinity of the space $\mathcal{N}(V_k)$ of *Hermitian norms* on $V_k := \Gamma(X, L^{\otimes k})$.

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$\mathcal{N}(V_k)$ is a symmetric space with a natural Riemannian metric $d_{2,k}$. Given $\|\cdot\| \in \mathcal{N}(V_k)$, and picking an orthogonal basis $(s_i)_i$, geodesic rays starting at $\|\cdot\|$ are of the form

$$\|\sum a_i s_i\|_t^2 = \sum |a_i|^2 \|s_i\|^2 e^{-2t\lambda_i}$$

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$$\left(\|\sum a_i s_i\|_t\right)^{1/t} \to_{t\to\infty} \max_i |a_i|_0 e^{-\lambda_i}$$

with $|a_i|_0 = 1$ iff $a_i \neq 0$ is the **trivial absolute value** on $\mathbb{C}$.

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This defines an **ultrametric** norm $\|\cdot\|^\infty$ on V_k, i.e. satisfying

$$\|\lambda s + t\|^\infty \leq \max(|\lambda|_0 \|s\|^\infty, \|t\|^\infty).$$

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To any such sequence of norms $\|\cdot\|_\bullet$ one can associate a compactly supported probability measure on $\mathbb{R}$, its **Duistermaat–Heckman measure** (Boucksom–Chen '08), which is a rescaled weak limit of

$$\sigma_k := \sum_i \delta_{\frac{-\log \|s_{i,k}\|_k}{k}}$$

with $(s_{i,k})_i$ a(n ultrametric) orthogonal basis for $\|\cdot\|_k$.

This allows us to state our main result. Let $\|\cdot\|_\bullet$ be an element in the boundary at infinity of $\mathcal{H}_L^\infty$, and let $\mathcal{C}$ be the space of **decreasing, bounded convex functions** on the support of the Duistermaat–Heckman measure of $\|\cdot\|_\bullet$.

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Theorem (R.-Witt Nyström '23)

For all $\phi \in \mathcal{H}_L^\infty$, there exists an isometric embedding

$$\iota : (\mathcal{C}, L^2) \hookrightarrow (\mathcal{H}_L^\infty, d_2)$$

mapping the apex of $\mathcal{C}$ to ϕ.

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Example.

Let $f_t(x) = -tx$. Then $t \mapsto \iota(f_t)$ is the canonical geodesic ray (in the sense of Phong–Sturm) starting at ϕ and directed by $\|\cdot\|_\bullet$.

Ideas involved it the construction.

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5. Use deep results of Finski '23 (analysing such geodesic rays) to show that flatness at $t = \infty$ (in the boundary at infinity) induces flatness at $t = 1$.