Submersion constructions for geometries with parallel skew torsion

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Some motivation

Let $G \subset \mathrm{O}(n)$, and consider a G-structure on a Riemannian manifold (M, g).
For our purposes, this is a section $\Phi \in \Gamma(\bigotimes TM)$ of some tensor bundle.

- The G-structure Φ is called *integrable* (or *torsion-free*), and (M, g, Φ) an *integrable special geometry* if $\nabla^g \Phi = 0$.
- This entails a holonomy reduction $\mathrm{Hol}(\nabla^g) \subset G \subset \mathrm{O}(n)$.
- Otherwise, it is called *non-integrable*, and $\nabla^g \Phi$ its *torsion*.
- Examples for integrable geometries: Kähler manifolds, torsion-free G_2-manifolds, ... (Berger's holonomy list)
- Sometimes related to the existence of parallel spinors.

Some motivation

- Nonintegrable G-structures are classified according to the algebraic type of their torsion (Gray–Hervella for $\mathrm{U}(n)$, Fernandez–Gray for G_2, ...)
- Most nonintegrable geometries do not have an integrable analogue!
- Example: Sasaki structures, where $G = \mathrm{U}(n) \times \{1\} \subset \mathrm{O}(2n+1)$.
- A particularly interesting class of nonintegrable G-structures are those for which there exists a G-connection with skew-symmetric torsion tensor.

Geometries with parallel skew torsion

Definition

A *geometry with parallel skew torsion* (M, g, τ) is a Riemannian manifold together with a 3-form $\tau \in \Omega^3(M)$ such that $\nabla^\tau \tau = 0$ for the connection on TM given by

$$\nabla^\tau := \nabla^g + \tau.$$

- $\tau_X := X \lrcorner \tau \in \Lambda^2 T^*M \cong \mathfrak{so}(TM)$ for any $X \in TM$.
- The torsion tensor is given by $T^\tau = 2\tau$.
- ∇^τ is a G-connection for the stabilizer group $G = \mathrm{Stab}(\tau) \subset \mathrm{O}(n)$, and $\mathrm{Hol}(\nabla^\tau) \subseteq G$.
- Geometries with parallel skew torsion are sometimes related to existence of Killing spinors.

Example: Gray manifolds

A *nearly Kähler manifold* (M, g, J) is an almost Hermitian manifold such that

$$\nabla^g J \in \Omega^3(M).$$

Its *canonical Hermitian connection* is given by

$$\nabla^\tau = \nabla^g + \tau, \quad \text{where} \quad \tau_X Y = -\frac{1}{2} J(\nabla^g_X J)Y.$$

This is a geometry with parallel skew torsion.
If $\dim M = 6$, $\tau \neq 0$ defines a nonintegrable $\mathrm{SU}(3)$-structure on (M, g), and we call (M, g, J) a *Gray manifold*.

Example: Nearly parallel G_2-manifolds

A G_2-manifold (M, g_φ, φ) is called *nearly parallel* if its torsion has only a scalar part, that is

$$d\varphi = \tau_0 \star_\varphi \varphi, \qquad \tau_0 \in \mathbb{R}.$$

Its *canonical G_2-connection* is given by

$$\nabla^\tau = \nabla^g + \tau, \quad \text{where} \quad \tau = \frac{\tau_0}{12}\varphi.$$

This is a geometry with parallel skew torsion.

Example: Naturally reductive spaces

Let $(G/H, \mathfrak{m})$ be a *reductive* homogeneous space, that is, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant decomposition.
Let g be an invariant Riemannian metric. $(G/H, \mathfrak{m}, g)$ is called *naturally reductive* if

$$g([X, Y]_{\mathfrak{m}}, Z) + g(Y, [X, Z]_{\mathfrak{m}}) = 0 \qquad \forall X, Y, Z \in \mathfrak{m}.$$

The *canonical reductive connection* is given by

$$\nabla^\tau = \nabla^g + \tau, \quad \text{where} \quad \tau_X Y = -\frac{1}{2}[X, Y]_{\mathfrak{m}}.$$

This is again a geometry with parallel skew torsion!

Holonomy

The following question has been asked in many contexts and variations:

Question (É. Cartan, de Rham, Berger, ...)

Can we classify geometries according to their holonomy?

More precisely, we would like to be able to:

- decompose geometries into "irreducible" ones,
- completely describe the irreducible geometries.

$\rightsquigarrow$ *"reduction to atoms"*.

Let us begin by reviewing a few theorems about Riemannian holonomy.

Review of Riemannian holonomy

Let (M^n, g) be a connected Riemannian manifold, let $\mathrm{Hol}(\nabla^g) \subset \mathrm{O}(n)$ be the Riemannian holonomy group, and $\mathrm{Hol}^0(\nabla^g)$ its connected component of the identity.

Theorem (deRham 1952, local version)

If at some point $p \in M$, the holonomy representation $\mathrm{Hol}^0(\nabla^g) \curvearrowright T_pM$ is reducible into $T_pM = \bigoplus_i T_i$, then (M, g) is locally isometric to a Riemannian product $\prod_i (M_i, g_i)$, and we have $\mathrm{Hol}^0(\nabla^g) = \prod_i \mathrm{Hol}^0(\nabla^{g_i})$.

Roughly: *Holonomy-reducible* $\Longleftrightarrow$ *locally product*.

Review of Riemannian holonomy

Let (M^n, g) be a connected Riemannian manifold, let $\mathrm{Hol}(\nabla^g) \subset \mathrm{O}(n)$ be the Riemannian holonomy group, and $\mathrm{Hol}^0(\nabla^g)$ its connected component of the identity.

Theorem (Berger 1955, Alekseevski 1968)

If (M^n, g) is not locally symmetric (i.e. $\nabla^g R^g \neq 0$) and has irreducible holonomy representation, then $\mathrm{Hol}^0(\nabla^g)$ is one of the following:

$$\mathrm{SO}(n), \quad \mathrm{U}(\tfrac{n}{2}), \quad \mathrm{SU}(\tfrac{n}{2}), \quad \mathrm{Sp}(1)\cdot\mathrm{Sp}(\tfrac{n}{4}), \quad \mathrm{Sp}(\tfrac{n}{4}),$$

$$\mathrm{Spin}(7) \quad (n=8), \qquad \mathrm{G}_2 \quad (n=7).$$

Decomposing skew torsion

Let (M, g, τ) be a geometry with parallel skew torsion, $\nabla^\tau = \nabla^g + \tau$.

- (M, g, τ) is called *decomposable* if there exists a nontrivial splitting $TM = T_1 \oplus T_2$ into ∇^τ-parallel distributions such that $\tau = \tau_1 + \tau_2$ for $\tau_i \in \Gamma(\Lambda^3 T_i)$. Otherwise, it is called *indecomposable*.

Theorem (Storm 2017, Cleyton–Moroianu–Semmelmann 2021)

If (M, g, τ) is decomposable, it is locally isometric to a product $(M_1, g_1, \tau_1) \times (M_2, g_2, \tau_2)$.

- Indecomposable geometries with parallel skew torsion may still be holonomy-reducible (for ∇^τ)!

The standard submersion

Let (M, g, τ) be a geometry with parallel skew torsion, $\nabla^\tau = \nabla^g + \tau$. The following construction is due to Cleyton–Moroianu–Semmelmann (2021).

- For $p \in M$, consider the representation of the holonomy Lie algebra $\mathfrak{hol}(\nabla^\tau) \subset \mathfrak{so}(n)$, and decompose it into irreducibles:

  $$T_pM = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k \oplus \mathfrak{v}_1 \oplus \ldots \oplus \mathfrak{v}_l,$$

  such that $\mathfrak{hol}(\nabla^\tau) \cap \mathfrak{so}(\mathfrak{h}_\alpha) \neq 0 \; \forall \alpha = 1, \ldots, k$, and $\mathfrak{hol}(\nabla^\tau) \cap \mathfrak{so}(\mathfrak{v}_j) = 0 \; \forall j = 1, \ldots, l$.
- Let $\mathcal{H}_\alpha, \mathcal{V}_j \subset TM$ be the associated ∇^τ-parallel distributions, and let $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ and $\mathcal{V} = \bigoplus_j \mathcal{V}_j$, respectively.

The standard submersion

(M, g, τ) geometry with parallel skew torsion, $\nabla^\tau = \nabla^g + \tau$, $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$, $\mathcal{V} = \bigoplus_j \mathcal{V}_j$, $TM = \mathcal{H} \oplus \mathcal{V}$.

- The torsion $\tau \in \Omega^3(M)$ splits as

$$\tau = \tau^{\mathcal{H}} + \tau^{\mathrm{m}} + \tau^{\mathcal{V}},$$

  where pointwise $\tau^{\mathcal{H}} \in \bigoplus_\alpha \Lambda^3 \mathcal{H}_\alpha$, $\tau^{\mathcal{V}} \in \Lambda^3 \mathcal{V}$, and $\tau^{\mathrm{m}} \in \bigoplus_\alpha \Lambda^2 \mathcal{H}_\alpha \otimes \mathcal{V}$.
- The distribution $\mathcal{V}$ is integrable, and thus defines a foliation of M.
- Near a point $p \in M$, let N be the local leaf space of this foliation.

The standard submersion

(M, g, τ) geometry with parallel skew torsion, $\nabla^\tau = \nabla^g + \tau$, $TM = \mathcal{H} \oplus \mathcal{V}$, $\tau = \tau^{\mathcal{H}} + \tau^{\mathrm{m}} + \tau^{\mathcal{V}}$.

- We obtain a *locally defined* Riemannian submersion $\pi : (M, g) \rightarrowtail (N, g_N)$ with totally geodesic fibers, the *standard submersion*.
- $\mathcal{H}$ and $\mathcal{V}$ are the *horizontal* and *vertical* distributions of π, respectively.
- $\tau^{\mathcal{H}}$ is projectable to N ($\tau^{\mathcal{H}} = \pi^*\sigma$ for $\sigma \in \Omega^3(N)$), and (N, g_N, σ) is again a geometry with parallel skew torsion.
- The fibers $(F, g\big|_F, \tau^{\mathcal{V}}\big|_F)$ of π are again geometries with parallel skew torsion and *parallel curvature* ($\nabla^\tau R^\tau = 0$), so they are *naturally reductive Ambrose-Singer manifolds*.
- The O'Neill invariant measuring the failure of $\mathcal{H}$ to be integrable is given by $A = -\tau^{\mathrm{m}}$.

The standard submersion

(M, g, τ) geometry with parallel skew torsion, $\nabla^\tau = \nabla^g + \tau$,
$\pi : (M, g, \tau) \rightarrowtail (N, g_N, \sigma)$ standard submersion.

To summarize:

- The fibers of π are "nice": they are (locally) naturally reductive homogeneous. Storm (2019) gave a classification scheme for these.
- Since $\tau \in \bigoplus_\alpha \Lambda^3 \mathcal{H}_\alpha$, the base (N, g_N, σ) is *decomposable*, thus locally a product $\prod_\alpha (N_\alpha, g_\alpha, \sigma_\alpha)$.
- Thus, we have "reduced" (M, g, τ) to smaller geometries.
- Yet we cannot infer that the $(N_\alpha, g_\alpha, \sigma_\alpha)$ are holonomy-irreducible.

Irreducible geometries with parallel skew torsion

Let (M, g, τ) be a geometry with parallel skew torsion, $\tau \neq 0$.

Theorem (Cleyton–Swann 2004, Moroianu–S. 2024)

If the representation of $\mathfrak{hol}(\nabla^\tau)$ on the tangent space is irreducible, then (M, g, τ) (locally) belongs to one of the following classes:

1. Non-symmetric isotropy irreducible homogeneous spaces.
2. The symmetric spaces $(G \times G)/G$, $G^{\mathbb{C}}/G$ or $(G \ltimes \mathfrak{g})/G$, where G is a compact simple Lie group.
3. Gray manifolds and weak holonomy G_2-manifolds.
4. $\dim M = 3$, with $\tau = c \cdot \mathrm{vol}_g$, $c \in \mathbb{R}$.

- Isotropy irreducible spaces are classified (Wolf 1968).
- The above result holds even without the assumption of τ being skew-symmetric.

Example: Twistor spaces over qK manifolds

Let (N^{4n}, g_N, ω_i), $i = 1, 2, 3$, be a quaternion-Kähler manifold, i.e. $\mathrm{Hol}(\nabla^{g_N}) \subseteq \mathrm{Sp}(1) \cdot \mathrm{Sp}(n)$, $\mathrm{span}\{\omega_1, \omega_2, \omega_3\} = \mathfrak{sp}(1)$.

- For any $p \in N$, denote with $Z_p \subset \Lambda^2 T_p M$ the 2-sphere of compatible complex structures on $T_p M$.
- The total space $Z := \coprod_{p \in N} Z_p$ is called the *twistor space* of (N, g_N), and $\pi : Z \to N$ is called the *twistor fibration*.

Theorem (Alexandrov–Grantcharov–Ivanov, 1998)

If g_N has positive scalar curvature, then there exist

1. a Kähler structure on Z,
2. a (strict) nearly Kähler structure on Z,

both making π into a Riemannian submersion with totally geodesic fibers.

Example: Sasaki manifolds

Let (M^{2n+1}, g, ξ, Φ) be a *Sasaki manifold*, that is, a Riemannian manifold equipped with $\xi \in \mathfrak{X}(M)$ and $\Phi \in \Omega^2(M)$ such that

$$d\xi = 2\Phi, \qquad \nabla^g_X \Phi = -2X \wedge \xi \qquad \forall X \in \mathfrak{X}(M).$$

Φ defines a complex structure on the distribution $\xi^\perp \subset TM$. $(M, g, \xi \wedge \Phi)$ is a geometry with parallel skew torsion.

Theorem (Boothby–Wang 1958, Hatakeyama 1963; local version)

There is a locally defined Riemannian submersion with totally geodesic fibers $(M^{2n+1}, g) \rightarrowtail (N^{2n}, g_N, \omega)$ over a Kähler manifold, such that $\pi^*\omega = \Phi$.

Example: 3-(α, δ)-Sasaki manifolds

A 3-(α, δ)-Sasaki manifold $(M^{4n+3}, g, \xi_i, \Phi_i)$ is a Riemannian manifold together with $\xi_i \in \mathfrak{X}(M)$, $\Phi_i \in \Omega^2(M)$, $i = 1, 2, 3$, $\alpha, \delta \in \mathbb{R}$, $\alpha \neq 0$, such that

$$\begin{aligned} \xi_k &= \xi_i \lrcorner \Phi_j = -\xi_j \lrcorner \Phi_i, \\ \Phi_k &= -\Phi_i \circ \Phi_j + \xi_j^\flat \otimes \xi_i = \Phi_j \circ \Phi_i - \xi_i^\flat \otimes \xi_j, \\ d\xi_i &= 2\alpha\Phi_i + 2(\alpha - \delta)\xi_j \wedge \xi_k \end{aligned}$$

for all $i, j, k = 1, 2, 3$ with $\operatorname{sign}(ijk) = 1$. This is again a geometry with parallel skew torsion, for the right choice of τ.

Theorem (Stecker 2021, local version)

There is a locally defined Riemannian submersion with totally geodesic fibers $\pi : (M^{4n+3}, g) \rightarrowtail (N^{4n}, g_N, \omega_i)$ over a quaternion-Kähler manifold, such that $\pi^*\omega_i = \Phi_i$.

Intermediate subalgebras

- You may have noticed that I did not mention the holonomy group at all in the last three examples.
- Indeed, these submersion constructions utilize the structure of the representation $\mathfrak{g} \curvearrowright T_pM$ for a fixed Lie algebra $\mathfrak{hol}(\nabla^\tau) \subseteq \mathfrak{g} \subseteq \mathfrak{stab}(\tau)$.

Theorem (Moroianu-S. 2024)

Let (M, g, τ) be a geometry with parallel skew torsion. We can replace $\mathfrak{hol}(\nabla^\tau)$ with $\mathfrak{g}$ in the definition of $TM = \mathcal{H} \oplus \mathcal{V}$ and call it the *canonical $\mathfrak{g}$-splitting*. This again gives a locally defined submersion $\pi : (M, g) \rightarrowtail (N, g_N)$, the *canonical $\mathfrak{g}$-submersion*, and it retains all the nice properties of the standard submersion.

The canonical $\mathfrak{g}$-submersion

- Making $\mathfrak{g}$ larger $\Longrightarrow$ making $\mathcal{V}$ smaller, and making $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ coarser.
- However, we now have a satisfying decomposition result:

Theorem (Moroianu-S. 2024)

The base (N, g_N, σ) of the canonical $\mathfrak{g}$-submersion is locally a product of geometries with parallel skew torsion $\prod_\alpha (N_\alpha, g_\alpha, \sigma_\alpha)$ which have *irreducible $\mathfrak{stab}(\sigma_\alpha)$-action*.

This begs two questions:

1. What are the $\mathfrak{stab}(\tau)$-irreducible geometries with parallel skew torsion?
2. Can we characterize the cases where $\mathfrak{hol}(\nabla^\tau) \subsetneq \mathfrak{stab}(\tau)$?

Irreducible geometries with parallel skew torsion II

Let (M, g, τ) be a geometry with parallel skew torsion, $\tau \neq 0$.

Theorem (Cleyton–Swann 2004, Moroianu–S. 2024)

If the representation of $\mathfrak{stab}(\tau)$ on the tangent space is irreducible, then (M, g, τ) (locally) belongs to one of the following classes:

1. Non-symmetric isotropy irreducible homogeneous spaces.
2. The symmetric spaces $(G \times G)/G$, $G^{\mathbb{C}}/G$ or $(G \ltimes \mathfrak{g})/G$, where G is a compact simple Lie group.
3. Gray manifolds and weak holonomy G_2-manifolds.
4. $\dim M = 3$, with $\tau = c \cdot \mathrm{vol}_g$, $c \in \mathbb{R}$.

These are the same classes as in the holonomy-irreducible case – but now we allow for holonomy reducible spaces!

Holonomy vs. stabilizer

We would like to compare $\mathfrak{hol}(\nabla^\tau)$ to $\mathfrak{stab}(\tau)$ in the classification theorem.

Theorem (Moroianu–S. 2024)

In cases ❶ (isotropy irreducible) and ❷ (symmetric of group type), the only spaces with $\mathfrak{hol}(\nabla^\tau) \subsetneq \mathfrak{stab}(\tau)$ are

- the Berger space $\mathrm{SO}(5)/\,\mathrm{SO}(3)_{\mathrm{irr}}$ with its nearly parallel G_2-structure, where $\mathfrak{hol}(\nabla^\tau) = \mathfrak{so}(3)_{\max} \subset \mathfrak{g}_2$,
- the compact and simple Lie groups $(G \times G)/G$ with the *flat* Cartan-$(\pm)$-connections.

Holonomy vs. stabilizer: Gray manifolds

Let (M, g, τ) be a Gray manifold: $\mathfrak{hol}(\nabla^\tau) \subseteq \mathfrak{stab}(\tau) = \mathfrak{su}(3)$.

Theorem (Moroianu–S. 2024)

We have $\mathfrak{hol}(\nabla^\tau) = \mathfrak{su}(3)$, except for

- the homogeneous Gray manifold $S^3 \times S^3 = \frac{S^3 \times S^3 \times S^3}{\mathrm{diag}(S^3)}$, where $\mathfrak{hol}(\nabla^\tau) = \mathfrak{so}(3) \subset \mathfrak{su}(3)$,
- complex reducible holonomy representation, i.e. $\mathfrak{hol}(\nabla^\tau) \subseteq \mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1))$. In this case (M, g, τ) is locally isometric to the *twistor space over a self-dual Einstein 4-manifold*.

Moreover, $\mathfrak{hol}(\nabla^\tau) \subsetneq \mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1))$ only for the homogeneous Gray manifold $\mathrm{SU}(3)/T^2$.

The complex reducible case is a converse to Reyes-Carrión (1993) and extends Belgun–Moroianu (2001).

Holonomy vs. stabilizer: Nearly parallel G_2 mfds.

Let (M, g, τ) be nearly parallel G_2: $\mathfrak{hol}(\nabla^\tau) \subseteq \mathfrak{stab}(\tau) = \mathfrak{g}_2$.

Theorem (Moroianu–S. 2024)

We have $\mathfrak{hol}(\nabla^\tau) = \mathfrak{g}_2$, except for

- the Berger space $\mathrm{SO}(5)/\mathrm{SO}(3)_{\mathrm{irr}}$ with its nearly parallel G_2-structure, where $\mathfrak{hol}(\nabla^\tau) = \mathfrak{so}(3)_{\max} \subset \mathfrak{g}_2$,
- reducible holonomy representation. In this case, (M, g, τ) is *3-(α, δ)-Sasakian* with $\delta = 5\alpha$, and $\mathfrak{hol}(\nabla^\tau) \subseteq \mathfrak{so}(4)$.

Moreover, $\mathfrak{hol}(\nabla^\tau) \subsetneq \mathfrak{so}(4)$ only if $\mathfrak{hol}(\nabla^\tau) = \mathfrak{u}(2)$ and the base (N^4, g_N) is Kähler–Einstein.

This extends Friedrich (2007). The holonomy-reducible case is a converse to Agricola–Dileo (2020).

Just for fun: Almost irreducible stabilizer action

Let (M, g, τ) be a geometry with parallel skew torsion.

Theorem (Moroianu–S. 2024)

Assume that the representation $\mathfrak{stab}(\tau) \curvearrowright T_pM$ is *almost irreducible* in the sense that

$$T_pM \cong \mathbb{R} \oplus V, \qquad V \text{ irreducible},$$

and that (M, g, τ) has no local 1-dimensional factor. Then $\mathfrak{stab}(\tau) = \mathfrak{u}(n)$, and (M^{2n+1}, g, τ) is *Sasakian*.

Moreover, if $\mathfrak{hol}(\nabla^\tau) \subsetneq \mathfrak{stab}(\tau)$, then the Kähler base (N^{2n}, g_N) of the canonical $\mathfrak{u}(n)$-submersion is *decomposable*, *positive Kähler–Einstein*, or *hyperkähler*.

Thank you!

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